On L-fuzzy approximation operators and L-fuzzy relations on residuated lattices

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Abstract

We consider properties of L-fuzzy relations and L-normal operators for a residuated lattice L in detail and show that the class RL(U) of all L-fuzzy relations on U and the class NL(U) of all L-normal operators are residuated lattices and they are isomorphic as lattices. Moreover, we prove that for any L-normal operator F, it is reflexive (or transitive) if and only if the L-fuzzy relation RF induced by F is reflexive (or transitive), respectively.

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1 Introduction

The rough set theory by Pawlak [9] has been actively researched as a valuable method for finding rules and features from incomplete data sets. The central concept of this theory is the notion of approximation space (U, E), where U is a non-empty finite set, and E is an equivalence relation on U. A subset of U divided by the equivalence relation E can be considered as representing some knowledge relative to U. When we extend this theory to a generalized approximation space (U, R), where U is (not necessarily finite) a set and R is (not necessarily equivalence) a binary relation on U, we face an essential problem that how we specify a subset representing rules or knowledge. As one of the methods to solve the problem, we use the approximation operator R (R), called the upper (lower) approximation operator induced by the binary relation R to determine the subset.

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representing knowledge about $U$. Since $\overline{R}(A)$ and $\underline{R}(A)$ for a subset $A \subseteq U$ is defined as follows:

\[
R(x) = \{ y \in U \mid (x, y) \in R \}, \\
\overline{R}(A) = \{ x \in U \mid R(x) \cap A \neq \emptyset \}, \\
\underline{R}(A) = \{ x \in U \mid R(x) \subseteq A \}.
\]

Moreover, these sets are considered as the data similar to the elements of $A$; it is possible to express pieces of knowledge, and rules in an incomplete data set. As to the generalized approximation spaces $(U, R)$, research on their topological properties \cite{1, 5, 6, 12} and algebraic properties \cite{4, 8, 14} are going on. Recently, for a mathematical structure, say a lattice $L$, research about $L$-fuzzy approximation spaces is also progressing. A map $A : U \to L$ is called an $L$-fuzzy set on $U$, and $R : U \times U \to L$ is said to be an $L$-fuzzy relation on $U$. For a lattice $L$, by an $L$-fuzzy approximation space, we mean the mathematical structure $(U, R)$, where $U$ is a non-empty set and $R$ is an $L$-fuzzy relation on $U$. The mathematical object $L$ can be selected such as distributive lattices, Boolean algebras and residuated lattices to the purposes. Research on $L$-fuzzy approximation spaces is one of the hot fields of rough set theory, and many papers have been published so far \cite{3, 7, 8, 13}. Since most of the proofs in such papers are element-based, it is a tedious work to check the conditions of the definitions one by one, and they are relatively long proofs. It is not easy to apply the results to other mathematical structures $L$.

In this paper, we consider $L$-fuzzy approximation spaces $(U, R)$ as operators on $L^U$ and $L^{U \times U}$ for a residuated lattice $L$, and provide operator-based proofs for their properties. Since the proofs are operator-based, they are relatively short and a good outlook. Therefore, the results can be easily applied to other cases. In order to give operator-based proofs of properties of $L$-fuzzy approximation spaces, we prepare the following definitions and basic properties.

Let $L$ be a residuated lattice, which definition is given later. For any $L$-fuzzy approximation space $(U, R)$, we define operators called upper (lower) approximation operator $\overline{R}(R)$, $\underline{R}(R) : L^U \to L^U$ as follows. For any $A \in L^U$,

\[
\overline{R}(A)(x) = \bigvee_{y \in U} (R(x, y) \odot A(y)), \\
\underline{R}(A)(x) = \bigwedge_{y \in U} (R(x, y) \to A(y)).
\]

From these operators, we consider an upper approximation $\overline{R}(A)$ (lower approximation $\underline{R}(A)$) of an $L$-fuzzy set $A$, respectively. This means, we can get operators $\overline{R}$, $\underline{R}$ on $L^U$ from the $L$-fuzzy relation $R$.

Conversely, the question of whether we can construct an $L$-fuzzy relation by an operator on $L^U$ arises naturally. However, this is trivially No!. Because if we consider the cardinalities of the set of $L$-fuzzy relations $R_L(U) = L^{U \times U}$ and that of the class of operators $(L^U)^{L^U}$ on $L^U$, then those classes do not have the same cardinality. So, there is no one-to-one correspondence between $L$-fuzzy relations and operators on $L^U$.

On the other hand, taking into account the properties of $L$-fuzzy relations on $U$, we have another interesting problem:

1. Under what conditions of operators on $L^U$ do we have an $L$-fuzzy relations from the operator?
2. If so, is the correspondence between such operators and the $L$-fuzzy relations one-to-one?
We give an affirmative answer to the problem in this paper. We also provide operator-based algebraic proofs to the results in [7] instead of original element-based proofs. This allows the above question to be treated more generally.

In addition, by considering $L$-fuzzy approximation spaces as a fuzzy-version of Kripke semantics in modal logic, the properties of $L$-fuzzy relation, such as reflexivity, symmetry, and transitivity, are easy to understand for using operators. This means, we have new knowledge about the relationship between $L$-fuzzy approximation spaces and modal logic.

2 Residuated lattices and fuzzy approximation spaces

Let $U$ be a non-empty set and $\mathcal{L} = \langle L, \land, \lor, \circ, 0, 1 \rangle$ be a complete residuated lattice, that is,

(i) $\langle L, \land, \lor, 0, 1 \rangle$ is a complete bounded lattice;

(ii) $\langle L, \circ, 1 \rangle$ is a commutative monoid;

(iii) For all $x, y, z \in L$,

$$x \circ y \leq z \iff x \leq y \rightarrow z.$$ 

For any element $x \in L$, we define $x' = x \rightarrow 0$. We denote a residuated lattice by its support set $L$ of $\mathcal{L}$ for the sake of simplicity. We have the following basic properties of residuated lattices [2].

**Proposition 2.1.** For all $x, y, z, x_i, y_i \in L$, we have

1. $0' = 1, 1' = 0$;
2. $x \circ x' = 0$;
3. $x \leq y \iff x \rightarrow y = 1$;
4. $x \circ (x \rightarrow y) \leq y$;
5. $x \leq y \implies x \circ z \leq y \circ z, z \rightarrow x \leq z \rightarrow y, y \rightarrow z \leq x \rightarrow z$;
6. $1 \rightarrow x = x$;
7. $x \lor (y \rightarrow z) \leq y \rightarrow (x \lor z)$;
8. $x \circ (\bigvee_i y_i) = \bigvee_i (x \circ y_i)$;
9. $(\bigvee_i x_i)' = \bigwedge_i x_i'$;
10. $x \rightarrow (\bigwedge_i y_i) = \bigwedge_i (x \rightarrow y_i), (\bigvee_i x_i) \rightarrow y = \bigwedge_i (x_i \rightarrow y)$.

A mapping $A : U \rightarrow L$ (i.e., $A \in L^U$) is simply called an $L$-fuzzy set on $U$. For any element $a \in L$, we define $L$-fuzzy sets $a$ and $a_x$ for $x \in U$ as follows:

$$a(x) = a \quad (\forall x \in U);$$

$$a_x(y) = \begin{cases} a & (y = x) \\ 0 & (y \neq x) \end{cases} \quad (\forall x, y \in U).$$
Let $1_S$ be the characteristic function of $S \subseteq U$. Thus, we have

$$1_{U-\{x\}}(y) = \begin{cases} 
1 & (y \neq x) \\
0 & (y = x) 
\end{cases} \quad (\forall y \in U).$$

We define an order $\leq$ on $L^U$ by the pointwise order, that is, for all $A, B \in L^U$,

$$A \leq B \text{ if and only if } A(x) \leq B(x) \text{ for all } x \in U.$$ 

Then $0$ and $1$ defined by

$$0(x) = 0 \quad (\forall x \in U),$$

$$1(x) = 1 \quad (\forall x \in U),$$

are the smallest and largest elements in $L^U$, respectively. Mathematical structures of $L^U$ inherit from those of $L$ as follows: For all $A, B, A_i \in L^U$, if we define

$$A'(x) = (A(x))', \quad \forall x \in U$$

$$(A \land B)(x) = A(x) \land B(x), \quad \forall x \in U$$

$$(A \lor B)(x) = A(x) \lor B(x), \quad \forall x \in U$$

$$(A \odot B)(x) = A(x) \odot B(x), \quad \forall x \in U$$

$$(A \rightarrow B)(x) = A(x) \rightarrow B(x), \quad \forall x \in U$$

$$\bigwedge_i A_i(x) = \bigwedge_i A_i(x), \quad \forall x \in U$$

$$\bigvee_i A_i(x) = \bigvee_i A_i(x), \quad \forall x \in U,$$

then $(L^U, \land, \lor, \odot, 0, 1)$ is also a complete residuated lattice. We note $1_{U-\{x\}} = (1_x)'$.

We recall some definitions about $L$-fuzzy relations. Let $U$ and $V$ be non-empty sets. In general, a map $R : U \times V \rightarrow L$ is called an $L$-fuzzy relation from $U$ to $V$. For the case of $U = V$, a map $R : U \times U \rightarrow L$ is simply called an $L$-fuzzy relation on $U$. An $L$-fuzzy relation $R$ on $U$ is also called

1. **reflexive** if $R(x,x) = 1 \quad (\forall x \in U)$;
2. **symmetric** if $R(x,y) = R(y,x) \quad (\forall x, y \in U)$;
3. **transitive** if $R(x,y) \odot R(y,z) \leq R(x,z) \quad (\forall x, y, z \in U)$;
4. **serial** if $\bigvee_{y \in U} R(x,y) = 1 \quad (\forall x \in U)$;
5. **Euclidean** if $R(x,y) \odot R(x,z) \leq R(y,z) \quad (\forall x, y, z \in U)$.

For a non-empty set $U$ and an $L$-fuzzy relation on $U$, a structure $(U, R)$ is called an $L$-fuzzy approximation space. According to [7, 9, 10, 11], we define an upper (lower) $L$-fuzzy approximation operators $\overline{R}$ $(\underline{R}) : L^U \rightarrow L^U$ as follows:

$$\overline{R}(A)(x) = \bigvee_{y \in U} (R(x,y) \odot A(y)),$$

$$\underline{R}(A)(x) = \bigwedge_{y \in U} (R(x,y) \rightarrow A(y)).$$
We mainly treat upper $L$-fuzzy approximation operators $\overline{R}$ in this paper.

By $\mathcal{R}_L(U)$, we mean the class of all $L$-fuzzy relations on $U$. We note that $\mathcal{R}_L(U)$ is the complete residuated lattice, because, since $\mathcal{R}_L(U) = L^U \times U$, it inherits properties from those of $L$.

**Proposition 2.2.** Let $R$ be an $L$-fuzzy relation on $U$. Then for all $a \in L$ and $A, B, A_i \in L^U$, we have

1. $\overline{R}(a \circ (\bigvee_i A_i)) = a \circ (\bigvee_i \overline{R}(A_i))$;
2. $\overline{R}(a \circ A) = a \circ \overline{R}(A)$;
3. $\overline{R}(\bigvee_i A_i) = \bigvee_i \overline{R}(A_i)$;
4. $\overline{R}(a) \leq a \leq \overline{R}(a)$;
5. $A \leq B \Rightarrow \overline{R}(A) \leq \overline{R}(B)$, $\overline{R}(A) \leq \overline{R}(B)$;
6. $\overline{R}(\bigwedge_i A_i) = \bigwedge_i \overline{R}(A_i)$;
7. $\overline{R}(a \vee A) \geq a \vee \overline{R}(A)$, hence $\overline{R}(a \vee (\bigwedge_i A_i)) \geq a \vee (\bigwedge_i \overline{R}(A_i))$;
8. $\overline{R}(a \rightarrow A) = a \rightarrow \overline{R}(A)$;
9. $a \circ \overline{R}(A) \leq \overline{R}(a \circ A)$.

**Proof.** We only prove the cases of (1) and (7). The other cases can be proved easily.

1. This is proved as follows.

\[
\overline{R}(a \circ (\bigvee_i A_i))(x) = \bigvee_{y \in U} \left( R(x, y) \circ (\bigvee_i a \circ (\bigvee_i A_i))(y) \right)
\]

\[
= \bigvee_{y \in U} \left( R(x, y) \circ a \circ \bigvee_i A_i(y) \right)
\]

\[
= a \circ \bigvee_{y \in U} \left( R(x, y) \circ \bigvee_i A_i(y) \right)
\]

\[
= a \circ \bigvee_{y \in U} \left( \bigvee_i R(x, y) \circ A_i(y) \right)
\]

\[
= a \circ \bigvee_i \left( \bigvee_{y \in U} R(x, y) \circ A_i(y) \right)
\]

\[
= a \circ \bigvee_i \overline{R}(A_i)(x)
\]

\[
= (a \circ (\bigvee_i \overline{R}(A_i)))(x).
\]

(7) Since $\overline{R}$ is an order preserving operator, it follows from (4) that

\[
a \vee \overline{R}(A) \leq R(a) \vee R(A) \leq R(a \vee A).
\]

$\square$
**Remark 2.3.** For (4), it follows that $\overline{R}(a) = a$ if and only if $\bigvee_{y \in U} R(x, y) = 1$ for all $x \in U$, that is, $R$ is serial.

We note that a pair of two results (2) and (3) is equivalent to (1). Moreover, the results (5), (6), (8) and (9) are obtained by a general result that $\overline{R}$ and $R^{-1}$ (and also $R^{-1}$ and $R$) forms an adjoint pair (denoted by $\overline{R} \dashv R^{-1}$), that is,

**Proposition 2.4.** For any $L$-fuzzy relation $R$ on $U$, we have

$$\overline{R}(A) \leq B \iff A \leq R^{-1}(B) \quad (\forall A, B \in L^U);$$

$$R^{-1}(A) \leq B \iff A \leq \overline{R}(B) \quad (\forall A, B \in L^U).$$

**Proof.** Since

$$\overline{R}(A) \leq B \iff (\overline{R}(A))(x) \leq B(x) \quad (\forall x \in U)$$
$$\iff \bigvee_{y \in U} (R(x, y) \odot A(y)) \leq B(x) \quad (\forall x \in U)$$
$$\iff R(x, y) \odot A(y) \leq B(x) \quad (\forall x, y \in U)$$
$$\iff A(y) \leq R(x, y) \rightarrow B(y) = R^{-1}(y, x) \rightarrow B(x) \quad (\forall x, y \in U)$$
$$\iff A(y) \leq \bigwedge_{x \in U} (R^{-1}(y, x) \rightarrow B(x)) \quad (\forall y \in U)$$
$$\iff A(y) \leq (R^{-1}(B))(y) \quad (\forall y \in U)$$
$$\iff A \leq R^{-1}(B),$$

a pair of two operators $\overline{R}$ and $R^{-1}$ forms the adjoint pair. Another case can be proved similarly if we take $R$ to be $R^{-1}$. \qed

For example, the result (9) $a \odot R(A) \leq R(a \odot A)$ can be proved as follows. Since $R^{-1} \vdash \overline{R}$, it is sufficient to show $R^{-1}(a \odot R(A)) \leq a \odot A$. It is obvious that $R^{-1}(a \odot A) = a \odot R^{-1}(R(A))$ and $R^{-1}(R(A)) \leq A$ by $R(A) \leq R(A)$. Therefore, we get

$$R^{-1}(a \odot R(A)) = a \odot R^{-1}(R(A)) \leq a \odot A.$$

**Corollary 2.5.** If $R$ is symmetric, then $\overline{R} \vdash R$.

Let $U$ and $V$ be non-empty sets. An operator $F : L^U \rightarrow L^V$ is called **normal** if it satisfies the condition:

$$F(a \odot \bigvee_{i} A_i) = a \odot \bigvee_{i} F(A_i) \quad (\forall a \in L, \forall A_i \in L^U).$$

It is easy to prove that

**Proposition 2.6.** For an operator $F : L^U \rightarrow L^V$, $F$ is normal if and only if it satisfies the conditions: For all $a \in L, A_i \in L^U$,

(N1) $F(a \odot A) = a \odot F(A)$;

(N2) $F(\bigvee_{i} A_i) = \bigvee_{i} F(A_i)$.

It follows from the Proposition 2.2 that
Corollary 2.7. For every L-fuzzy relation \( R \) on \( U \), the operator \( \overline{R} \) is normal.

It is clear to show the next result.

Proposition 2.8. For all normal operators \( F, G : L^U \to L^U \), the composition operator \( F \circ G \) defined by \( (F \circ G)(A) = F(G(A)) \) \( (\forall A \in L^U) \) is a normal operator.

For any map \( \varphi : U \to V \), the Zadeh’s fuzzy backward operator (simply backward operator) \( \lceil \), \( \varphi^- : L^V \to L^U \) is defined by

\[
\varphi^-(B)(x) = B(\varphi(x)) \quad (\forall B \in L^V, \forall x \in U).
\]

Proposition 2.9. For every map \( \varphi : U \to V \), the backward operator \( \varphi^- : L^V \to L^U \) is normal.

Proof. We show that for any \( x \in U, b \in L, B, B_i \in L^V, \)

\[
(1) \quad \varphi^-(b \circ B)(x) = b \circ \varphi^-(B) \quad \text{and} \quad \text{(2) } \varphi^-(\bigvee_i B_i) = \bigvee_i \varphi^-(B_i).
\]

For the case (1), we have the following sequence of equations.

\[
\varphi^-(b \circ B)(x) = (b \circ B)(\varphi(x)) = b(\varphi(x)) \circ B(\varphi(x))
\]

\[
= b \circ (\varphi^-(B))(x) = b(x) \circ (\varphi^-(B))(x)
\]

\[
= (b \circ \varphi^-(B))(x).
\]

Therefore, we get \( \varphi^-(b \circ B)(x) = b \circ \varphi^-(B). \)

As to the case (2), we also have

\[
\varphi^-(\bigvee_i B_i)(x) = (\bigvee_i B_i)(\varphi(x)) = \bigvee_i (B_i(\varphi(x)) = \bigvee_i (\varphi^-(B_i))(x) = \bigvee_i \varphi^-(B_i)(x),
\]

and thus \( \varphi^-(\bigvee_i B_i) = \bigvee_i B_i. \)

Therefore, the backward operator \( \varphi^- \) is normal.

In order to show the fundamental and essential property about normal operators, we need the following lemma \([11]\).

Lemma 2.10. For any L-fuzzy set \( A \in L^U \), \( A = \bigvee_{x \in U}(A(x) \circ 1_x) \), where \( A(x) \) and \( 1_x \) are defined respectively by

\[
A(x)(t) = A(x) \in L \quad (\forall x \in U) \quad \text{and} \quad 1_x(t) = \begin{cases} 1 & (t = x) \\ 0 & (\text{otherwise}) \end{cases}.
\]

For two operators \( F, G : L^U \to L^V \), a partial order \( \leq \) is defined as usual,

\[
F \leq G \text{ if and only if } F(A) \leq G(A) \text{ for all } A \in L^U.
\]

Now, we prove the fundamental and important property about normal operators. It says that the partial order \( \leq \) on normal operators are determined only by the element \( 1_x \in L^U \) for all \( x \in U. \)

Theorem 2.11. For two normal operators \( F, G : L^U \to L^V \), we have

\[
F \leq G \text{ if and only if } F(1_x) \leq G(1_x) \text{ for all } x \in U.
\]
Proof. It is sufficient to show that \( \mathcal{F} \leq \mathcal{G} \) if \( \mathcal{F}(1_x) \leq \mathcal{G}(1_x) \) for all \( x \in U \).

Let \( A \) be arbitrary element in \( L^U \). Since \( A = \bigvee_{x \in U} (A(x) \odot 1_x) \), we have

\[
\mathcal{F}(A) = \mathcal{F} \left( \bigvee_{x \in U} (A(x) \odot 1_x) \right) = \bigvee_{x \in U} \mathcal{F}(A(x) \odot 1_x) \quad (\because \mathcal{F} \text{ is normal})
\]
\[
= \bigvee_{x \in U} (A(x) \odot \mathcal{F}(1_x)) \quad (\because \mathcal{F} \text{ is normal})
\]
\[
\leq \bigvee_{x \in U} (A(x) \odot \mathcal{G}(1_x)) = \mathcal{G} \left( \bigvee_{x \in U} (A(x) \odot 1_x) \right) \quad (\because \mathcal{G} \text{ is normal})
\]
\[
= \mathcal{G}(A).
\]

Therefore, we get \( \mathcal{F}(A) \leq \mathcal{G}(A) \) for all \( A \in L^U \). This means that \( \mathcal{F} \leq \mathcal{G} \).

\[ \square \]

Corollary 2.12. For normal operators \( \mathcal{F}, \mathcal{G} \),

\[ \mathcal{F} = \mathcal{G} \text{ if and only if } \mathcal{F}(1_x) = \mathcal{G}(1_x) \text{ for all } x \in U. \]

Now we consider properties of \( L \)-fuzzy relation from \( U \) to \( V \). Let \( R \) and \( S \) be \( L \)-fuzzy relation from \( U \) to \( V \). We define a partial order \( \subseteq \) on the set of all \( L \)-fuzzy relations from \( U \) to \( V \) as follows:

\[ R \subseteq S \iff R(x, y) \leq S(x, y) \quad (\forall x \in U, y \in V). \]

Proposition 2.13. For any \( L \)-fuzzy relation \( R, S \) from \( U \) to \( V \), we have

\[ R \subseteq S \text{ if and only if } \overline{R} \leq \overline{S}. \]

Proof. Suppose \( R \subseteq S \), that is, \( R(x, y) \leq S(x, y) \) for all \( x \in U, y \in V \). For all \( B \in L^V \) and \( x \in U \), since

\[
(\overline{R}(B))(x) = \bigvee_{y \in V} (R(x, y) \odot B(y))
\]
\[
\leq \bigvee_{y \in V} (S(x, y) \odot B(y)) = (\overline{S}(B))(x),
\]

we have \( \overline{R}(B) \leq \overline{S}(B) \) for all \( B \in L^V \) and thus \( \overline{R} \leq \overline{S} \).

Conversely, we assume \( \overline{R} \leq \overline{S} \), namely, \( \overline{R}(B) \leq \overline{S}(B) \) for all \( B \in L^V \). Let \( x \in U, y \in V \). If we take \( 1_y \in L^V \) as \( B \in L^V \), that is, \( \overline{R}(1_y) \leq \overline{S}(1_y) \), then

\[
(\overline{R}(1_y))(x) \leq (\overline{S}(1_y))(x).
\]

By definition of \( \overline{R} \), we have

\[
(\overline{R}(1_y))(x) = \bigvee_{t \in V} (R(x, t) \odot 1_y(t)) = R(x, y) \odot 1_y(y) = R(x, y).
\]

Similarly, \( (\overline{S}(1_y))(x) = S(x, y) \). It follows that

\[
R(x, y) \leq S(x, y) \quad (\forall x \in U, y \in V).
\]

This means that \( R \subseteq S \).

\[ \square \]

Corollary 2.14. For any \( L \)-fuzzy relations \( R, S \) on \( U \), i.e. \( R, S : U \times U \to L \), \( R = S \) if and only if \( \overline{R} = \overline{S} \).
3 Fuzzy natural transformation

Let \((X, R)\) and \((Y, S)\) be two \(L\)-fuzzy approximation spaces. In [7], two important notions about \(L\)-fuzzy approximation spaces are defined and studied. A one-to-one map \(\varphi : X \to Y\) is called an upper fuzzy backward natural transformation from \((X, R)\) to \((Y, S)\) if
\[
\mathcal{R}(\varphi^{-}(B)) \leq \varphi^{-}(\mathcal{S}(B)), \quad (\forall B \in L^{Y}).
\]
It is also represented by \(\mathcal{R} \circ \varphi^{-} \leq \varphi^{-} \circ \mathcal{S}\) in operator-based notation.

A map \(\varphi : X \to Y\) is called relation preserving if
\[
R(x, y) \leq S(\varphi(x), \varphi(y)), \quad (\forall x, y \in U).
\]

The following result is proved in [7]:

**Proposition 4.1** Let \((X, R)\) and \((Y, S)\) be two fuzzy approximation spaces and \(\varphi : X \to Y\) be a one-to-one map. Then \(\varphi\) is an upper fuzzy backward natural transformation if and only if \(\varphi\) is relation preserving.

The result is proved by an element-based method, so it has a long proof and also has few generalizations. We here provide operator-based proof about it. Our proof makes the result to apply to more wide cases. We prepare some results to do so.

At first, we note that
\[
R(x, y) = \mathcal{R}(1_{y})(x) \quad \text{and} \quad S(\varphi(x), \varphi(y)) = \varphi^{-}(\mathcal{S}(1_{\varphi(y)}))(x), \quad (\forall x, y \in U).
\]
So, a relation preserving map \(\varphi : U \to V\) from \((U, R)\) to \((V, S)\) can be represented by
\[
\mathcal{R}(1_{u}) \leq \varphi^{-}(\mathcal{S}(1_{\varphi(u)})), \quad (\forall u \in U).
\]

**Lemma 3.1.** For any map \(\varphi : U \to V\), we have
\[
1_{u} \leq \varphi^{-}(1_{\varphi(u)}), \quad (\forall u \in U).
\]
Moreover, if \(\varphi\) is injective (one-to-one), then
\[
1_{u} = \varphi^{-}(1_{\varphi(u)}), \quad (\forall u \in U).
\]

**Proof.** It follows from \(\{t \in U \mid u = t\} \subseteq \{t \in U \mid \varphi(u) = \varphi(t)\}\) that \(1_{u}(t) \leq \varphi^{-}(1_{\varphi(u)})(t)\) for all \(t \in U\) and thus
\[
1_{u} \leq \varphi^{-}(1_{\varphi(u)}), \quad (\forall u \in U).
\]
Moreover, if \(\varphi\) is injective, since \(\{t \in U \mid t = u\} = \{t \in U \mid \varphi(u) = \varphi(t)\}\), then we get \(1_{u}(t) = \varphi^{-}(1_{\varphi(u)})(t)\) for all \(t \in U\), thus
\[
1_{u} = \varphi^{-}(1_{\varphi(u)}), \quad (\forall u \in U).
\]

\[\square\]
Lemma 3.2. Let $\varphi : U \to V$ be a map and $R, S$ are $L$-fuzzy relations on $U$ and on $V$, respectively. Then we have

1. $(R \circ \varphi^\frown)(B) = (R \circ \varphi^\frown)(B^*)$ \quad ($\forall B \in L^V$), where $B^*$ is defined by $B^*(v) = \begin{cases} B(v) & (v \in \varphi(U)) \\ 0 & (v \notin \varphi(U)) \end{cases}$,

2. $(\varphi^\frown \circ \overline{S})(B^*) \leq (\varphi^\frown \circ \overline{S})(B)$, \quad ($\forall B \in L^V$).

Proof. For the case (1), let $B \in L^V$ and $x \in U$. Since 

\[
(R \circ \varphi^\frown)(B)(x) = \bigvee_{y \in U} (R(x, y) \odot (\varphi^\frown B)(y)) \\
= \bigvee_{y \in U} (R(x, y) \odot B(\varphi(y))) \\
= \bigvee_{y \in U} (R(x, y) \odot B^*(\varphi(y))) \\
= (R \circ \varphi^\frown)(B^*)(x),
\]

we get $(R \circ \varphi^\frown)(B) = (R \circ \varphi^\frown)(B^*)$ for all $B \in L^V$.

The case (2) can be proved similarly. \hfill \Box

Now, we provide an operator-based proof to the result in [7] above.

Theorem 3.3. Let $(X, R)$ and $(Y, S)$ be two fuzzy approximation spaces and $\varphi : X \to Y$ a one-to-one map. Then $\varphi$ is a (upper) fuzzy backward natural transformation if and only if $\varphi$ is relation preserving, that is,

$\varphi$ is relation preserving $\iff R \circ \varphi^\frown \leq \varphi^\frown \circ \overline{S}$.

Proof. \((\Rightarrow)\) We assume that a one-to-one map $\varphi$ is relation preserving, that is, $R(1_u) \leq \varphi^\frown(\overline{S}(1_{\varphi(u)}))$ for all $u \in U$. Since the operators $R \circ \varphi^\frown$ and $\varphi^\frown \circ \overline{S}$ are both normal, in order to show $R(\varphi^\frown(B)) \leq \varphi^\frown(\overline{S}(B))$ \quad ($\forall B \in L^Y$), it is sufficient to show by Lemma 3.2 that

\[
(R \circ \varphi^\frown)(1_{\varphi(u)}) \leq (\varphi^\frown \circ \overline{S})(1_{\varphi(u)}) \quad (\forall u \in U).
\]

Since $\varphi$ is the one-to-one map, we have $R(1_u) = R(\varphi^\frown(1_{\varphi(u)}))$. It follows from assumption that

\[
(R \circ \varphi^\frown)(1_{\varphi(u)}) = R(\varphi^\frown(1_{\varphi(u)})) = R(1_u) \quad (\because \varphi \text{ is one-to-one}) \\
\leq (\varphi^\frown \circ \overline{S})(1_{\varphi(u)}) \quad (\text{by assumption})
\]

\((\Leftarrow)\) Conversely, we suppose that $R \circ \varphi^\frown \leq \varphi^\frown \circ \overline{S}$. Since $\varphi$ is one-to-one, we have $1_u = \varphi^\frown(1_{\varphi(u)})$ and

$R(1_u) = R(\varphi^\frown(1_{\varphi(u)})) = (R \circ \varphi^\frown)(1_{\varphi(u)}) \leq (\varphi^\frown \circ \overline{S})(1_{\varphi(u)})$.

Therefore, $\varphi$ is the relation preserving map. \hfill \Box
4 Normal operators and \( L \)-fuzzy relations

In the last section, we consider a relation between the class \( \mathcal{N}_L(U) \) of all normal operators on \( U \) and the class \( \mathcal{R}_L(U) \) of all \( L \)-fuzzy relations on \( U \). We show that there is a one-to-one correspondence between them, and these classes are isomorphic as lattices.

Let \( U \) be a non-empty set and \( \mathcal{F} : L^U \rightarrow L^U \) be an operator. We define a (upper) fuzzy transformation system according to \([7]\). A structure \((U, \mathcal{F})\) is called a (upper) fuzzy transformation system if

1. \( A(x) \leq \mathcal{F}(A)(x) \), \( \forall A \in L^U, \ x \in U \);
2. \( \mathcal{F}(\bigvee_i A_i) = \bigvee_i \mathcal{F}(A_i) \), \( \forall \{A_i \mid i \in I\} \subseteq L^U \);
3. \( \mathcal{F}(a \odot A) = a \odot \mathcal{F}(A) \), \( \forall a \in L, A \in L^U \);

Remark 4.1. In our definition of a fuzzy transformation system is different from the original one in \([7]\). In \([7]\), the following condition

4. \( \text{core}(\mathcal{F}(1_x)) \neq \emptyset \), where \( \text{core}(A) = \{x \mid A(x) = 1, x \in U\} \),

is assumed for the definition of fuzzy transformation systems. However, the condition can be obtained from the condition (1) as follows.

Since \( 1_x \leq \mathcal{F}(1_x) \) by (1) and \( 1_x(x) = 1 \), we have \( 1 = 1_x(x) \leq \mathcal{F}(1_x)(x) \) and thus \( 1 = \mathcal{F}(1_x)(x) \). This means that \( x \in \text{core}(\mathcal{F}(1_x)) \) and \( \text{core}(\mathcal{F}(1_x)) \neq \emptyset \). Therefore, the original condition (4) above is redundant.

Moreover, an operator \( \mathcal{F} : L^U \rightarrow L^U \) is called reflexive if \( A \leq \mathcal{F}(A) \) for all \( A \in L^U \). Therefore, the notion of fuzzy transformation systems \( \mathcal{F} : L^U \rightarrow L^U \) is precisely the same as that of reflexive normal operators.

We show that there is a one to one correspondence between the normal operators and the \( L \)-fuzzy relations on \( U \). At first, we treat a general case.

Theorem 4.2. Let \( U, V \) be non-empty sets. For any normal operator \( \mathcal{F} : L^V \rightarrow L^U \), there exists a unique \( L \)-fuzzy relation \( R : U \times V \rightarrow L \) from \( U \) to \( V \) such that \( \mathcal{F} = R \).

Proof. For any normal operator \( \mathcal{F} : L^V \rightarrow L^U \), we define \( R : U \times V \rightarrow L \) by

\[ R(x, y) = \mathcal{F}(1_y)(x), \ \forall x \in U, \ y \in V. \]

It is clear that \( R(1_y) = \mathcal{F}(1_y) \) for all \( y \in V \). Since \( R \) ad \( \mathcal{F} \) are normal, we get

\[ R = \mathcal{F}. \]

The uniqueness is proved as follows. If \( \overline{R} = \mathcal{F} = \overline{S} \) for two \( L \)-fuzzy relations \( R \) and \( S \), then we have \( \overline{R} = \overline{S} \) and thus \( R = S \) by Corollary 2.14.

In any fuzzy transformation system \((U, \mathcal{F})\), since \( \mathcal{F} \) is the reflexive normal operator, the result holds immediately.

Corollary 4.3 (Theorem 5.1 in \([7]\)). For any (upper) fuzzy transformation system \((U, \mathcal{F})\), that is, \( \mathcal{F} : L^U \rightarrow L^U \), there exists a unique \( L \)-fuzzy relation \( \overline{R} \) on \( U \) such that \( \mathcal{F} = \overline{R} \).
Now we consider the relation between normal operators and L-fuzzy relations. For an operator \( \mathcal{F} : L^U \rightarrow L^U \), an L-fuzzy relation \( R_\mathcal{F} : U \times U \rightarrow \mathcal{L} \) is defined by

\[
R_\mathcal{F}(x, y) = (\mathcal{F}(1_y))(x), \quad (\forall x, y \in U).
\]

Conversely, for an L-fuzzy relation \( R : U \times U \rightarrow \mathcal{L} \), we define an operator \( \mathcal{F}_R : L^U \rightarrow L^U \) by \( \mathcal{F}_R = \overline{R} \), that is,

\[
(\mathcal{F}_R(A))(x) = \bigvee_{y \in U} (R(x, y) \circledast A(y)), \quad (\forall x \in U).
\]

**Theorem 4.4.** Let \( \mathcal{F} : L^U \rightarrow L^U \) be a normal operator and \( R : U \times U \rightarrow \mathcal{L} \) an L-fuzzy relation.

1. \( \mathcal{F} = \overline{R} \)
2. \( R = R(\mathcal{F}_R) \)
3. \( R \sqsubseteq S \) iff \( \mathcal{F}_R \leq \mathcal{F}_S \) (i.e. iff \( \overline{R} \leq \overline{S} \)).

Therefore, the class \( \mathcal{N}_L(U) \) of all normal operators on \( U \) and the class \( \mathcal{R}_L(U) \) of all L-fuzzy relations on \( U \) both form lattices and they are isomorphic:

\[
\mathcal{N}_L(U) \cong \mathcal{R}_L(U).
\]

**Proof.**

(1) Since \( R_\mathcal{F} : U \times U \rightarrow \mathcal{L} \) is an L-fuzzy relation, the operator \( \overline{R_\mathcal{F}} \) is normal. For all \( x, y \in U \), we have

\[
(\overline{R_\mathcal{F}}(1_y))(x) = \bigvee_{t \in U} (R_\mathcal{F}(x, t) \circledast 1_t(t)) = R_\mathcal{F}(x, y) = (\mathcal{F}(1_y))(x) \quad (\forall x \in U).
\]

and \( \overline{R_\mathcal{F}}(1_y) = \mathcal{F}(1_y) \) for all \( y \in U \). Since \( \overline{R_\mathcal{F}}, \mathcal{F} \) are normal, this means that \( \mathcal{F} = \overline{R_\mathcal{F}} \).

(2) For all \( x, y \in U \), we get

\[
(\overline{R(\mathcal{F}_R)}(1_y))(x) = \bigvee_{t \in U} (R(\mathcal{F}_R)(x, t) \circledast 1_y(t)) = R(\mathcal{F}_R)(x, y) = (\mathcal{F}_R(1_y))(x) = (\overline{R}(1_y))(x),
\]

and \( \overline{R(\mathcal{F}_R)}(1_y) = \overline{R}(1_y) \). The fact that \( \overline{R}, \overline{R(\mathcal{F}_R)} \) are both normal operators implies \( \overline{R(\mathcal{F}_R)} = \overline{R} \) and \( R = R(\mathcal{F}_R) \).

(3) Proposition 2.13

It follows from the above that a map \( \xi : \mathcal{N}_L(U) \rightarrow \mathcal{R}_L(U) \) defined by

\[
\xi(\mathcal{F}) = R_\mathcal{F}, \quad (\forall \mathcal{F} \in \mathcal{N}_L(U)),
\]

gives a lattice isomorphism between \( \mathcal{N}_L(U) \) and \( \mathcal{R}_L(U) \). \( \square \)

Let \( \mathcal{F} : L^U \rightarrow L^U \) be a normal operator. For a normal operator \( \mathcal{F} \), it is called reflexive if \( A \leq \mathcal{F}(A) \) for all \( A \in L^U \). The following results are proved in \([11]\). However, we provide the proofs using the normality property, that is, the operator \( \overline{R} \) or \( R_\mathcal{F} \) is determined by only \( 1_x \) for all \( x \in U \).

**Proposition 4.5.** \([11]\) Let \( \mathcal{F} \) be a normal operator \( \mathcal{F} : L^U \rightarrow L^U \). Then

\( \mathcal{F} \) is reflexive if and only if \( R_\mathcal{F} \) is reflexive.
Proof. Let \( \mathcal{F} \) be reflexive. Since
\[
R_{\mathcal{F}}(x, x) = R_{\mathcal{F}}(1_x)(x) = \mathcal{F}((1_x))(x) \geq 1_x(x) = 1,
\]
we have \( R_{\mathcal{F}}(x, x) = 1 \) for all \( x \in U \), that is, \( R_{\mathcal{F}} \) is reflexive.

Conversely, suppose that \( R_{\mathcal{F}} \) is reflexive. It is sufficient to show that \( 1_x \leq \mathcal{F}(1_x) \) for all \( x \in U \), because \( 1 \) and \( \mathcal{F} \) are normal. Since \( R_{\mathcal{F}} \) is reflexive, we have
\[
1_x(t) \leq R_{\mathcal{F}}(t, x) = \mathcal{F}(1_x)(t), \quad (\forall t, x \in U),
\]
and \( 1_x \leq \mathcal{F}(1_x) \) for all \( x \in U \). This means that the operator \( \mathcal{F} \) is reflexive. \( \square \)

A normal operator \( \mathcal{F} : L^U \to L^U \) is called transitive if \( \mathcal{F}(\mathcal{F}(A)) \leq \mathcal{F}(A) \) for all \( A \in L^U \).

Theorem 4.6. [11] Let \( \mathcal{F} : L^U \to L^U \) be a normal operator. Then
\( \mathcal{F} \) is transitive if and only if \( R_{\mathcal{F}} \) is transitive.

Proof. We suppose that \( \mathcal{F} \) satisfies \( \mathcal{F}(\mathcal{F}(A)) \leq \mathcal{F}(A) \) for all \( A \in L^U \). Since \( \mathcal{F}(\mathcal{F}(1_z)) \leq \mathcal{F}(1_z) \), for all \( z \in U \) and \( \mathcal{F} = R_{\mathcal{F}} \), we have
\[
R_{\mathcal{F}}(R_{\mathcal{F}}(1_z))(x) \leq R_{\mathcal{F}}(1_z)(x), \quad (\forall x \in U).
\]
The facts that
\[
R_{\mathcal{F}}(R_{\mathcal{F}}(1_z))(x) = \bigvee_{y \in U} (R_{\mathcal{F}}(x, y) \circ (R_{\mathcal{F}}(1_z)(y))) = \bigvee_{y \in U} (R_{\mathcal{F}}(x, y) \circ R_{\mathcal{F}}(y, z)),
\]
and
\[
R_{\mathcal{F}}(1_z)(x) = R_{\mathcal{F}}(x, z),
\]
imply
\[
\bigvee_{y \in U} (R_{\mathcal{F}}(x, y) \circ R_{\mathcal{F}}(y, z)) \leq R_{\mathcal{F}}(x, z).
\]
Therefore, \( R_{\mathcal{F}} \) is transitive.

Conversely, let \( R_{\mathcal{F}} \) be transitive, that is, for all \( x, y, z \in U \)
\[
R_{\mathcal{F}}(x, y) \circ R_{\mathcal{F}}(y, z) \leq R_{\mathcal{F}}(x, z).
\]
Taking into account of
\[
(\mathcal{F}(1_z))(x) = (R_{\mathcal{F}}(1_z))(x) = R_{\mathcal{F}}(x, z),
\]
we have
\[
R_{\mathcal{F}}(R_{\mathcal{F}}(1_z))(x) = \bigvee_{y \in U} (R_{\mathcal{F}}(x, y) \circ (R_{\mathcal{F}}(1_z)(y)))
\]
\[
= \bigvee_{y \in U} (R_{\mathcal{F}}(x, y) \circ R_{\mathcal{F}}(y, z))
\]
\[
\leq R_{\mathcal{F}}(x, z) = R_{\mathcal{F}}(1_z)(x),
\]
and \( R_{\mathcal{F}}(R_{\mathcal{F}}(1_z)) \leq R_{\mathcal{F}}(1_z) \), in other words, \( \mathcal{F}(\mathcal{F}(1_z)) \leq \mathcal{F}(1_z) \). It follows from normality that \( \mathcal{F} \) is the transitive operator. \( \square \)
We mainly consider properties of $\overline{R}$ for $L$-fuzzy relation $R$ so far. On the other hand, if we think about the properties of $R$ then the following result can be applicable that there is a one to one correspondence between $\overline{R}$ and operators $\mathcal{H} : L^U \to L^U$ such that for all $a \in L$, $A, A_i \in L^U$

$$\mathcal{H}(a \to A) = a \to \mathcal{H}(A),$$

$$\mathcal{H}(\bigwedge_i A_i) = \bigwedge_i \mathcal{H}(A_i).$$

Therefore, it is similarly proved that for any operator $\mathcal{H} : L^U \to L^U$ satisfying the conditions above, there exists a unique $L$-fuzzy relation $R$ on $U$ such that $\mathcal{H} = \overline{R}$.

Moreover, since our proof so far is mainly operator-based proof, not element-based one, other properties of $L$-fuzzy relations of $R$ on $U$ proved above can be represented by using $\overline{R}$ and $R$ as follows. This comes from the idea of Kripke semantics for modal logic.

**Theorem 4.7.** [11] Let $R$ be an $L$-fuzzy relation on $U$. Then we have

1. $R$ is symmetric if and only if $A \leq R(\overline{R}(A))$, for all $A \in L^U$;
2. $R$ is serial if and only if $R(1_x) = 1$, for all $x \in U$;
3. $R$ is Euclidean if and only if $R(1_x) \leq R(\overline{R}(1_x))$, for all $x \in U$.

For a normal operator $\mathcal{F} : L^U \to L^U$, if it is reflexive and transitive, then it satisfies

(i) $\mathcal{F}(0) = 0$;
(ii) $A \leq \mathcal{F}(A)$, $(\forall A \in L^U)$;
(iii) $\mathcal{F}(\mathcal{F}(A)) = \mathcal{F}(A)$, $(\forall A \in L^U)$;
(iv) $\mathcal{F}(\bigvee_i A_i) = \bigvee_i \mathcal{F}(A_i)$, $(\forall A_i \in L^U)$.

This means that a reflexive and transitive normal operator is a closure operator on $L^U$, therefore,

$$\tau = \{ \mathcal{F}(A) \mid A \in L^U \},$$

forms an $L$-fuzzy topology [11]. Moreover it is an Alexandrov topology, that is, $\bigvee_i A_i$ is also closed for every $L$-fuzzy closed set $A_i \in \tau$.

**Theorem 4.8.** Every reflexive and transitive normal operator induces an Alexandrov topology on $L^U$.

5 Conclusions

In this paper, we consider properties of $L$-fuzzy relations and $L$-normal operators on a residuated lattice $L$ by operator-based and prove that there is a one-to-one correspondence between the class of all $L$-fuzzy relations and the class of all $L$-normal operators. We also give a simple and general proof to the result (Theorem 3.3) that for $L$-fuzzy approximation spaces $(X, R)$ and $(Y, S)$ and a map $\varphi : X \to Y$, $\varphi$ is a (upper) fuzzy backward natural transformation if and only if $\varphi$ is relation preserving. We do not need the injectiveness of the map $\varphi$, which is needed in the original result in [7]. Moreover, we prove that for any $L$-normal operator $\mathcal{F}$, it is reflexive (or transitive) if and only if the $L$-fuzzy relation $R_\mathcal{F}$ induced by $\mathcal{F}$ is reflexive (or transitive), respectively.
Declarations

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