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Hausdorff (quasi)topological MV-algebras

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Abstract

In this paper, the relations between separation axioms and (quasi)topological MV-algebras are studied. It is proved that T_0-spaces and (T_1) Hausdorff spaces are equivalent in (quasi)topological MV-algebras. Also, some topologies on MV-algebras are generated by ideals, filters and prefilters. It is shown that the MV-algebras equipped with these topologies are (para)topological MV-algebras and (T_0) normal spaces. In addition, some conditions are derived for locally compact Hausdorff MV-algebras to make them into normal paratopological MV-algebras. Finally, quotient MV-algebras are studied to get a Hausdorff topological quotient MV-algebra.

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1 Introduction

Topology and algebra are two fundamental areas of mathematics that play complementary roles. Topology studies continuity and convergence and provides a general framework to study the concept of a limit. Algebra considers all kinds of operations and provides a basis for algorithms and calculations. In recent decades, topology and algebra are widely used in the study of logic. The combined investigation of logic and other mathematical branches, such as algebra, topology and so on, promotes the development of logic and also enriches the content of these mathematical branches.

In recent years, many mathematicians have endowed some of the algebraic structures associated with logical systems with a topology. For instance, Roudabri and Torkzadeh [17] used the left (right) stabilizers of a BCK-algebra and produced two bases for two different topologies. Ahn and Kwon [1] studied topological properties in BCC-algebras. Borzooei et al. [4, 5] defined semitopological and topological BL-algebras and explored separation axioms on (semi)topological quotient BL-algebras. Yang et al. [20] (Zahiri and Borzooei [21]) constructed a topology on EQ-algebras (BL-algebras) using a system of filters. Furthermore, Khanegir et al. [12] studied uniform topology on BL-algebras. The concept of linear topology on IL-algebras was introduced by Islam et al. [11]. Chang [7] developed an algebraic version of Łukasiewicz logic and provided an algebraic proof of the completeness. The resulting algebraic system was known as an MV-algebra.

Undoubtedly, MV-algebras are among the most important structures associated with logical systems. MV-algebras stand concerning to the Łukasiewicz infinite-valued logic as Boolean algebras stand concerning to classical 2-valued logic. Of course, MV-algebras have not stayed glued to their origin in logic, and their applications have been shown in other areas of mathematics. For example, Hoo [10] introduced linear and I-adic topologies on MV-algebras by ideals and studied various properties of them. Weber [18] proved that the topology of a topological MV-algebra is uniquely determined by its neighborhood system of 0. Later, Nganou [16] studied strongly complete MV-algebras and gave a characterization of Stone topological MV-algebras. Najafi et al. [15] introduced (semi, para, quasi)topological MV-algebras and investigated neighborhood systems of 0 and 1 in topological MV-algebras. Further, they studied the continuity of auxiliary operations in (quasi)topological MV-algebras. Recently, some researchers applied proper filters of an MV-algebra to construct topological MV-algebras (see [3, 13, 14, 19]).

In this paper, we investigate the separation axioms on MV-algebras endowed with a topology. In Section 3, we prove that any T_0 quasitopological MV-algebra is a T_1-space, and any T_0 topological MV-algebra is a Hausdorff space. In Section 4, we introduce some topologies on MV-algebras by prefilters, filters and ideals to obtain (regular) normal (para)topological MV-algebras. Locally compact Hausdorff MV-algebras are studied in Section 5. We provide some conditions under which a topological quotient MV-algebra becomes a T_1-space (Hausdorff space) in Section 6.

2 Preliminaries

In this section, we recall some definitions and results on topological spaces, MV-algebra theory, and (semi, quasi)topological MV-algebras to make this paper self-contained and easy to read.

Definition 2.1. [6] *Consider the topological space (A, τ). We have the following separation axioms:*
(i) T_0*: For each $x, y \in A$ and $x \neq y$, there exists at least one in an open neighborhood excluding the other.*
(ii) T_1*: For each $x, y \in A$ and $x \neq y$, each has an open neighborhood not containing the other.*
(iii) T_2*: For each $x, y \in A$ and $x \neq y$, both have disjoint open neighborhoods U and V such that $x \in U$ and $y \in V$.*
(iv) *Regular: For each $x \in U \in \tau$, there exists an open set H such that $x \in H \subseteq \overline{H} \subseteq U$.*
(v) *Normal: For each closed set S and each open set U containing S, there is an open set H such that $S \subseteq H \subseteq \overline{H} \subseteq U$.*

A topological space satisfying T_i is called a T_i-space, for $i = 0, 1, 2$. A T_2-space is also known as a Hausdorff space. A topological space (A, τ) is said to be compact, if each open covering of A is reducible to a finite open covering, locally compact, if for each $x \in A$ there exists an open neighborhood U of x and a compact subset K such that $x \in U \subseteq K$. Also (A, τ) is said to be disconnected if there are two non-empty, disjoint, and open subsets $U, V \subseteq A$ such that $A = U \cup V$, connected otherwise, and totally disconnected if each non-empty connected subset of A has only one point.

Definition 2.2. [6] *Let $\mathcal{U}_x$ denotes the totality of all neighborhoods of x in A. Then, a subfamily $\mathcal{V}_x$ of $\mathcal{U}_x$ is said to form a fundamental system of neighborhoods of x, if for each U_x in $\mathcal{U}_x$, there exists V_x in $\mathcal{V}_x$ such that $V_x \subseteq U_x$.*

Definition 2.3. [2] *The family ξ of non-empty subsets of a set X is called a prefilter on X if $X \in \xi$ and for elements $A_1, ..., A_k$ of ξ, there exists $B \in \xi$ such that $B \subseteq \cap_{i=1}^{k} A_i$.*

Definition 2.4. [9] *An MV-algebra is an algebra $(A, \oplus, *, 0)$ of type (2,1,0) such that for any $x, y \in A$, the following conditions hold:*
(MV_1) $(A, \oplus, 0)$ *is an abelian monoid,*
(MV_2) $x \oplus 0^* = 0^*$,
(MV_3) $(x^*)^* = x$,
(MV_4) $(x^* \oplus y)^* \oplus y = (x \oplus y^*)^* \oplus x$.

Definition 2.5. [9] *Let A be an MV-algebra. For any $x, y \in A$ the constant 1 and the operations $\odot, \ominus, \rightarrow$ are defined as follows:*
(MV_5) $1 := 0^*$,
(MV_6) $x \odot y := (x^* \oplus y^*)^*$,
(MV_7) $x \ominus y := x \odot y^*$,
(MV_8) $x \rightarrow y := (x \odot y^*)^*$.

In any MV-algebra A, for any $x, y \in A$, $x \leq y$ if and only if $x^* \oplus y = 1$. The relation $\leq$ is a partial order relation on A, which determines a structure of distributive lattice, where the join $x \vee y = y \oplus (x \ominus y)$, the meet $x \wedge y = x \odot (x^* \oplus y)$, 0 is the smallest element and 1 is the biggest element [8]. By (MV_6) and (MV_7), for any $x, y \in A$, $x \leq y \Longleftrightarrow x \ominus y = 0$.

Example 2.6. [7] *Let S be a subset of the unit interval $I = [0, 1]$. If for any $x, y \in S$, $x \oplus y = min(1, x + y)$ and $x^* = 1 - x$, then $(S, \oplus, *, 0)$ is an MV-algebra. The following sets are examples of this type of MV-algebras.*
(i) $S = \{0, 1\}$ *which is called the trivial MV-algebra.*
(ii) $S = [0, 1]$ *which is called the standard MV-algebra.*
(iii) $S_n = \{\frac{m}{n} : m \in \{0, 1, 2, ..., n\}\}$, *where $n \in \mathbb{N}$.*

Proposition 2.7. [7] *Let A be an MV-algebra. The following properties hold for any $x, y \in A$:*
(M_1) $(A, \odot, 1)$ *is an abelian monoid,*
(M_2) $x \oplus x^* = 0^* = 1$,
(M_3) $x \oplus 0 = x \odot 1 = x$,
(M_4) *if* $x \oplus y = 0$, *then* $x = y = 0$,
(M_5) *if* $x \odot y = 1$, *then* $x = y = 1$,
(M_6) *if* $x \oplus y^* = y \oplus x^* = 1$, *then* $x = y$,
(M_7) *if* $x \ominus y = y \ominus x = 0$, *then* $x = y$,
(M_8) $x \odot 0 = x \ominus x = x \odot x^* = 0$.

Definition 2.8. [9] *Let A be an MV-algebra.*
(i) *A non-empty subset F of A is called a filter if F is closed with respect to $\odot$ and $x \leq y$, $x \in F$ imply $y \in F$.*
(ii) *A non-empty subset I of A is called an ideal if I is closed with respect to $\oplus$ and $x \leq y$, $y \in I$ imply $x \in I$.*

Proposition 2.9. [8] *Let I be an ideal of MV-algebra A. Then the binary relation $\stackrel{I}{\equiv}$ on A defined by $x \stackrel{I}{\equiv} y \Leftrightarrow x \ominus y \in I$ and $y \ominus x \in I$, is a congruence relation on A, i.e. it is an equivalence relation on A such that for any $a, b, c, d \in A$ if $a \stackrel{I}{\equiv} b$ and $c \stackrel{I}{\equiv} d$, then $a \oplus c \stackrel{I}{\equiv} b \oplus d$ and $a^* \stackrel{I}{\equiv} b^*$.*

Also, let $\frac{x}{I} = \{y \in A : x \overset{I}{\equiv} y\}$ *be an equivalence class of* x *and* $\frac{A}{I} = \left\{\frac{x}{I} : x \in A\right\}$. *Then* $\frac{A}{I}$ *is an MV-algebra under the following operations*

$$\frac{x}{I} \oplus \frac{y}{I} = \frac{x \oplus y}{I}, \quad (\frac{x}{I})^* = \frac{x^*}{I}.$$

Definition 2.10. [8] *Let* I *be an ideal of MV-algebra* A. $(\frac{A}{I}, \oplus, *, \frac{0}{I})$ *is called the quotient MV-algebra. Moreover, the correspondence* $x \longrightarrow \frac{x}{I}$ *defines the homomorphism* $\pi_I : A \longrightarrow \frac{A}{I}$, *which is called the natural homomorphism from* A *onto* $\frac{A}{I}$.

Definition 2.11. [15] *Let* A *be an MV-algebra with a topology* τ. *Then* (A, τ) *is called a:*
(i) *semitopological MV-algebra, if* $\oplus$ *is semicontinuous, equivalently, if for any* $a, x \in A$ *and any open neighborhood* U *of* $a \oplus x$, *there exists an open neighborhood* V *of* x *such that* $a \oplus V \subseteq U$,
(ii) *paratopological MV-algebra, if the operation* $\oplus$ *is continuous, equivalently, if for any* $x, y \in A$ *and any open neighborhood* W *of* $x \oplus y$, *there exist two open neighborhoods* U *and* V *of* x *and* y, *respectively, such that* $U \oplus V \subseteq W$,
(iii) *quasitopological MV-algebra, if the operation* $\oplus$ *is semicontinuous and the operation* $*$ *is continuous,*
(iv) *topological MV-algebra, if the operations* $\oplus$ *and* $*$ *are continuous.*

Proposition 2.12. [15] *If* (A, τ) *is a topological MV-algebra, then the mapping* $*(x) = x^*$ *from* A *into* A *is a homeomorphism.*

Proposition 2.13. [15] *Let* (A, τ) *be a topological MV-algebra and* I *be an ideal of* A. *If* 0 *is an interior point of* I, *then* I *is an open set.*

Example 2.14. [15] (i) *Let* S *be the standard MV-algebra and* τ *be the subspace topology of* $\mathbb{R}$. *Then* (S, τ) *is a topological MV-algebra.*
(ii) *If* $S_4 = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ *and* $\tau = \{\emptyset, \{0\}, \{0, \frac{1}{4}\}, S_4\}$, *then* (S_4, τ) *is a paratopological MV-algebra.*

Notation: Let A be an MV-algebra and $a \in A$. We define the following maps from A into A.
(i) $T_a(x) = a \oplus x$,
(ii) $L_a(x) = a \ominus x$,
(iii) $R_a(x) = x \ominus a$,
(iv) $D_a(x) = x \odot a$.
From now on, in this paper, (A, τ) is a topological space where A is an MV-algebra.

3 T_i-(quasi)topological MV-algebras

In this section, we determine the conditions that (A, τ) is a T_i-space $(i = 0, 1, 2)$. We prove that in quasitopological MV-algebras, any T_0-space is a T_1-space and in topological MV-algebras, T_0-spaces and T_2-spaces are equivalent. These results can be used to show that a topological space (A, τ) is not a (quasi)topological MV-algebra.

Proposition 3.1. *Let* R_a, L_a *or* D_a (T_a) *be an open map for any* $a \in A$. *If there exists* $U \in \tau$ *containing 1 (0), then* (A, τ) *is a* T_0*-space.*

Proof. Let $x \neq y$ and for any $a \in A$, R_a be an open map. If U is an open set containing 1, then $R_{x^*}(U) = U \ominus x^*$ and $R_{y^*}(U) = U \ominus y^*$ are open sets. By (M_3) and (MV_3), we have

$$1 \ominus x^* = 1 \odot (x^*)^* = x, \ 1 \ominus y^* = 1 \odot (y^*)^* = y.$$

Hence, $x \in R_{x^*}(U)$ and $y \in R_{y^*}(U)$. We claim that $y \notin U \ominus x^*$ or $x \notin U \ominus y^*$. Let $y \in U \ominus x^*$ and $x \in U \ominus y^*$. Then for some $a \in U,\ y = a \ominus x^*$. By (M_8),

$$y \odot x^* = a \odot x \odot x^* = a \odot 0 = 0 \Longrightarrow y \leq x.$$

Similarly, $x \in U \ominus y^*$ follows $x \leq y$. Therefore, $x = y$ which is a contradiction. Hence (A, τ) is a T_0-space. The proof is similar for other maps. □

Lemma 3.2. [2] *Let $(X, *)$ be a monoid with identity 1, τ be a topology on X, and $\mathcal{U} = \{U\}$ be a fundamental system of open neighborhoods of 1 in X. If (X, τ) is a T_1-space, then $\bigcap_{U \in \mathcal{U}} U = \{1\}$.*

Theorem 3.3. *Let (A, τ) be a quasitopological MV-algebra. Then the following are equivalent.*
(i) (A, τ) is a T_0-space.
(ii) (A, τ) is a T_1-space.
(iii) $\bigcap_{U \in \mathcal{U}} U = \{1\}$, where $\mathcal{U}$ is a fundamental system of open neighborhoods of 1.
(iv) For any $x \neq 1$, there are open neighborhoods U and V of x and 1, respectively, such that $1 \notin U$ and $x \notin V$.

Proof. $(i) \Rightarrow (ii)$ Let (A, τ) be a T_0-space and $x \neq y \in A$. By (M_7) $x \ominus y \neq 0$ or $y \ominus x \neq 0$. If $x \ominus y \neq 0$, then there exists an open set U such that $0 \in U$ and $x \ominus y \notin U$ or $0 \notin U$ and $x \ominus y \in U$. Suppose that $0 \in U$ and $x \ominus y \notin U$. By (M_8) $x \ominus x = y \ominus y = 0 \in U$. Since (A, τ) is a quasitopological MV-algebra, the operation $\ominus$ is semicontinuous [15]. Therefore, there exist open neighborhoods W and V of x and y, respectively, such that $V \ominus y \subseteq U$ and $x \ominus W \subseteq U$. We claim that $x \notin V$ and $y \notin W$. If $x \in V$ or $y \in W$, then $x \ominus y \in U$, which is a contradiction.
$(ii) \Rightarrow (iii)$ The proof follows from Lemma 3.2.
$(iii) \Rightarrow (iv)$ Let $x \neq 1$. Then there exists an open neighborhood U of 1 such that $x \notin U$. Since $x \to x = 1$ and (A, τ) is a quasitopological MV-algebra, there exists an open neighborhood V of x such that $V \to x \subseteq U$. We claim $1 \notin V$. If $1 \in V$, then $1 \to x = x \in U$, which is a contradiction.
$(iv) \Rightarrow (i)$ Let for any $x \neq 1$, there exist open neighborhoods U and V of x and 1, respectively, such that $1 \notin U$ and $x \notin V$. Suppose that $x, y \in A$ and $x \neq y$. Then $x \oplus y^* \neq 1$ or $y \oplus x^* \neq 1$. Let $x \oplus y^* \neq 1$ and U be an open neighborhood of $x \oplus y^*$ such that $1 \notin U$. Since (A, τ) is a quasitopological MV-algebra, there are two open neighborhoods V and W of x and y^*, respectively, such that $V \oplus y^* \subseteq U$ and $x \oplus W \subseteq U$. We claim that $x \notin W^*$ and $y \notin V$. If $x \in W^*$ or $y \in V$, then $y \oplus y^* = x \oplus x^* = 1 \in U$, which is a contradiction. Hence (A, τ) is a T_0-space. □

Example 3.4. (*i*) *Consider the MV-algebra $S_2 = \{0, \frac{1}{2}, 1\}$ with topology $\tau = \{S_2, \emptyset, \{0\}, \{1\}, \{0, 1\}\}$. Although the operation $*$ is continuous, the operation $\oplus$ is not semicontinuous at $(\frac{1}{2}, \frac{1}{2})$ which implies (S_2, τ) is not a quasitopological MV-algebra. This also follows from Theorem 3.3 because (S_2, τ) is a T_0-space while it is not a T_1-space.*
(ii) Consider the MV-algebra $S_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ with topology $\tau = \{S_3, \emptyset, \{1\}, \{0, \frac{1}{3}, \frac{2}{3}\}\}$. The case (iv) of Theorem 3.3 holds, but (S_3, τ) is not a T_0-space. The reason is that (S_3, τ) is not a quasitopological MV-algebra.

Theorem 3.5. *Let (A, τ) be a topological MV-algebra. Then (A, τ) is a T_0-space if and only if it is a Hausdorff space.*

Proof. Let (A, τ) be a topological MV-algebra and T_0-space. Since (A, τ) is a quasitopological MV-algebra, by Theorem 3.3, it suffices to show that if (A, τ) is a T_1-space, then it is a Hausdorff space. Let (A, τ) be a T_1-space and $x \neq y$. By (M_6), for any $x \neq y \in A$, $x \oplus y^* \neq 1$ or $x^* \oplus y \neq 1$.

If $x \oplus y^* \neq 1$, then there exists an open set U such that $x \oplus y^* \in U$ and $1 \notin U$. Since the operations $\oplus$ and $*$ are continuous, there exist open sets W and V of x and y^*, respectively, such that $W \oplus V \subseteq U$. W and V^* are open neighborhoods of x and y, respectively. We claim that $W \cap V^* = \emptyset$. If $z \in W \cap V^*$, then $z^* \in V$ and so $1 = z \oplus z^* \in W \oplus V \subseteq U$. Since the map $*$ is open by Proposition 2.12, W and V^* are disjoint open neighborhoods of x and y, respectively. Hence (A, τ) is a Hausdorff space. The proof of the converse is straightforward. □

Example 3.6. *Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. Consider the operations $\oplus$ and $*$ as follows:*

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

Then $(A, \oplus, *, 0)$ is an MV-algebra.
(i) If $\tau = \{A, \emptyset, \{a, 0\}, \{b, 1\}\}$, then (A, τ) is a topological MV-algebra [15]. *Since (A, τ) is not a Hausdorff space, it is not a T_0-space, by Theorem 3.5. Also, we have a fundamental system of neighborhoods $\mathcal{W}$ of 1 such that $\bigcap \mathcal{W} = \{b, 1\} \neq \{1\}$, as we expected by Theorem 3.3.*
(ii) Let $\tau = \{A, \emptyset, \{a\}, \{b\}, \{b, 1\}, \{a, 0\}, \{a, b\}, \{a, b, 1\}, \{a, b, 0\}\}$. (A, τ) is a T_0-space but it is not a Hausdorff space. Therefore, it is not a topological MV-algebra by Theorem 3.5.

Theorem 3.7. *Let (A, τ) be a topological MV-algebra.*
(i) (A, τ) is a Hausdorff space if and only if $\{1\}$ is a closed set.
(ii) (A, τ) is a discrete space if and only if $\{0\}$ is an open set.

Proof. (i) Let $\{1\}$ be a closed set. We show that $\{a\}$ is closed for any $a \in A$. Since the operation $\ominus$ is continuous [15], $B = \ominus^{-1}(\{1\}) = \{(1, 1)\}$ is a closed set. The function $g_a : A \longrightarrow A \times A$ defined by $b \longrightarrow (T_{a^*}(b), T_a(b^*))$ is also continuous. Then $g_a^{-1}\{(1, 1)\}$ is a closed set. On the other hand, by (M_6) we have

$$g_a^{-1}\{(1, 1)\} = \{b : b \oplus a^* = a \oplus b^* = 1\} = \{a\}.$$

Therefore, $\{a\}$ is a closed set. Hence (A, τ) is a T_1-space and so a Hausdorff space by Theorem 3.5. Since single sets are closed in Hausdorff spaces, the converse is clear.
(ii) The proof is similar to (i). □

Example 3.8. *Consider $S_3 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ with topology $\tau = \{S_3, \emptyset, \{0, \frac{1}{3}, \frac{2}{3}\}, \{1, \frac{1}{3}, \frac{2}{3}\}, \{\frac{1}{3}, \frac{2}{3}\}\}$. Since $\{1\}$ is a closed set and (S_3, τ) is not a Hausdorff space, (S_3, τ) is not a topological MV-algebra by Theorem 3.7.*

Proposition 3.9. *Let (A, τ) be a topological MV-algebra.*
(i) If F is a filter of A, then $\overline{F}$ is also a filter.
(ii) If I is an ideal of A, then $\overline{I}$ is also an ideal.

Proof. (i) Let (A, τ) be a topological MV-algebra and F be a filter of A. Since the operation $\odot$ is continuous, $\overline{F} \odot \overline{F} \subseteq \overline{F \odot F}$. Let $x, y \in A$ and $x \in \overline{F}$ such that $x \leqslant y$. There exists a net $\{x_j : j \in J\}$ in F which converges to x. By continuity of $\vee$, the net $\{x_j \vee y : j \in J\}$ converges to $x \vee y = y$. This implies that there are two cases:
Case 1. There exists $n \in \mathbb{N}$ such that $\forall j > n$, $x_j \leq y$. Then $y \in F \subseteq \overline{F}$.
Case 2. $\{x_j : j \in J\}$ converges to y. It follows that $y \in \overline{F}$.
(ii) The proof is similar to (i). □

The following example shows that the converse of Proposition 3.9 is not true in general.

Example 3.10. *Let (A, τ) be the topological MV-algebra in Example 3.6(i). Then $\overline{\{b\}} = \{b, 1\}$ is a filter but $\{b\}$ is not a filter. Also, $\overline{\{a\}} = \{0, a\}$ is an ideal while $\{a\}$ is not an ideal.*

Theorem 3.11. *Let (A, τ) be a topological MV-algebra. If the only closed filters (ideals) of A are $\{1\}$ ($\{0\}$) and A, then the topology τ is one of the following types:*
(i) Hausdorff and totally disconnected,
(ii) Hausdorff and connected,
(iii) indiscrete.

Proof. We prove this theorem in the case of filters. The proof for ideals is similar. Let C be the connected component of 1. It follows that C is a closed filter of A [15]. Hence $C = A$ or $C = \{1\}$. If $C = \{1\}$, by Theorem 3.7, (A, τ) is a Hausdorff space. If $x \in A$, and C_x is the connected component containing x, then $C_x \oplus x^*$ is connected and contains 1. Hence, $C_x \oplus x^* \subseteq C = \{1\}$. Then, for any $y \in C_x$, we have $y \oplus x^* = 1$, and so $x \leqslant y$. Since the operation $*$ is a homeomorphism, then $x \oplus C_{x^*}$ is connected and contains 1. Therefore, $y \leqslant x$ which implies that $x = y$ and so $C_x = \{x\}$. Hence, (A, τ) is a Hausdorff and totally disconnected space. Now, let $C = A$. By Proposition 3.9, $\overline{\{1\}}$ is a closed filter. Hence $\overline{\{1\}} = \{1\}$ or $\overline{\{1\}} = A$. In the first case, (A, τ) is a Hausdorff and connected space by Theorem 3.7(i). In the second case, let U is an arbitrary non-empty open subset of A. Thus $U \cap \{1\} \neq \emptyset$ and so $1 \in U$. Moreover, let $x \notin U$, for some $x \in A$. Since $x \oplus x^* = 1 \in U$, there exist open neighborhoods V and W of x and x^*, respectively, such that $V \oplus W^* \subseteq U$. Then, $x = x \oplus 0 \in U$, which is a contradiction. Therefore, if $x \in A$, then $x \in U$ which implies (A, τ) is an indiscrete space. □

Proposition 3.12. *Let (A, τ) be a topological MV-algebra. Then (A, τ) is an Uryshon space if and only if for any $x \neq 0$, there exist two open neighborhoods U and V of x and 0, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$.*

Proof. If (A, τ) is an Uryshon space, then the proof is clear. Conversely, let for any $x \neq 0$, there exist two open neighborhoods U and V of x and 0, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. If $x \neq y$, then $x \ominus y \neq 0$ or $y \ominus x \neq 0$. Let $x \ominus y \neq 0$. Hence there exist disjoint open neighborhoods U and V of $x \ominus y$ and 0, respectively, such that $\overline{U} \cap \overline{V} = \emptyset$. Since (A, τ) is a topological MV-algebra, there exist two open neighborhoods W_1 and W_2 of x and y, respectively, such that $W_1 \ominus W_2 \subseteq U$. We prove that $\overline{W}_1$ and $\overline{W}_2$ are disjoint. Let $z \in \overline{W}_1 \cap \overline{W}_2$. Hence there exist two nets $\{x_j : j \in J\}$ and $\{y_j : j \in J\}$ in W_1 and W_2, respectively, such that both converge to z. Since the operation $\ominus$ is continuous, the net $\{x_j \ominus y_j : j \in J\}$ converges to $z \ominus z = 0$. Therefore, $0 \in \overline{W_1 \ominus W_2} \subseteq \overline{U}$, which is a contradiction. □

4 Normal and regular MV-algebras

In this section, we construct some topologies on MV-algebras by prefilters, filters, and ideals. Then we determine the conditions that the MV-algebras equipped with these topologies become regular or normal spaces. By some examples, we show that these topologies exist and they are not trivial.

Theorem 4.1. *Let $\mathcal{F}$ be a prefilter on the MV-algebra A such that*
(i) $0 \in \bigcap \mathcal{F}$,
(ii) if $R_q \circ R_p(x) = 0$ for $p, q \in V \in \mathcal{F}$, then $x \in V$.
Then there exists a topology τ on A such that (A, τ) is a topological MV-algebra and normal space.

Proof. For an arbitrary element $a \in A$ and $\emptyset \neq V \subseteq A$, we define

$$V(a) = \{x \in A : R_a(x) \in V,\ L_a(x) \in V\}.$$

Clearly, if $V \subseteq U \subseteq A$, then $V(a) \subseteq U(a)$. If we put $\tau = \{U \subseteq A : \forall a \in U, \exists V \in \mathcal{F}\ s.t.\ V(a) \subseteq U\}$, then (A, τ) is a topological MV-algebra [15]. Now, we show that (A, τ) is a normal space. For this purpose, we prove that for any $x \in A$ and $V \in \mathcal{F}$, $V(x)$ is a closed set. If $y \in \overline{V(x)}$, then there exists $z \in V(x) \cap V(y)$. Hence $x, z \in V(x) \subseteq V$ and $y, z \in V(y) \subseteq V$. Since $((x \ominus y) \ominus x) \ominus z = 0$ and $((y \ominus x) \ominus y) \ominus x = 0$, then $x \ominus y \in V$ and $y \ominus x \in V$ by (ii). It follows that $y \in V(x)$ and so $\overline{V(x)} = V(x)$. Let S be a closed set and U be an open set such that $S \subseteq U$. For $x \in S$, there exists $V \in \mathcal{F}$ such that $V(x) \subseteq U$. By putting $H(x) = \bigcup_{x \in S, V \in \mathcal{F}} V(x)$, it follows that H is a closed and open set. Then $S \subseteq H \subseteq \overline{H} \subseteq U$ which implies (A, τ) is a normal space. □

Example 4.2. *Let A be the MV-algebra in Example 3.6. Put $\mathcal{F} = \{\{0, b\}, \{0\}, \{a, 0\}, A\}$. One can readily show that $\mathcal{F}$ is a prefilter. Consider the topology $\tau = \{A, \{0, b\}, \{1, a\}, \emptyset\}$. By Theorem 4.1, (A, τ) is a topological MV-algebra and normal space. Also, (A, τ) is not an Uryshon space by Proposition 3.12.*

Theorem 4.3. *Let I be a non-trivial ideal of the MV-algebra A. Then there exists a non-trivial topology τ such that I is a clopen set and (A, τ) is a*
(i) paratopological MV-algebra and T_0-space,
(ii) regular space if and only if for any $x \in U \in \tau$, $I \oplus x$ is a closed set,
(iii) normal space if and only if for any closed set S, $\bigcup_{x \in S} (I \oplus x)$ is a closed set.

Proof. Let I be a non-trivial ideal of A. Put $\tau = \{U \subseteq A : \forall x \in U,\ I \oplus x \subseteq U\}$. It is easy to show that τ is a topology on A. First, we prove that for any $a \in A$, $a \oplus I$ is an open set. Let $a \in A$ and $y \in a \oplus I$. Then

$$I \oplus y \subseteq I \oplus (a \oplus I) = (I \oplus I) \oplus a \subseteq I \oplus a.$$

Therefore, $a \oplus I$ is an open set and τ is a non-trivial topology on A. Since for $x \in I$, $x \oplus I \subseteq I$, then I is an open set. For $x \in \overline{I}$, $(I \oplus x) \cap I \neq \emptyset$. Therefore, there exists $a \in I$ such that $a \oplus x \in I$, and so $x \in I$ which implies that I is a closed set.
(i) Let $x \oplus y \in A$ and U be an open neighborhood of $x \oplus y$. Hence $x \oplus y \oplus I \subseteq U$. Since $x \oplus I$ and $y \oplus I$ are open neighborhoods of x and y, respectively, such that

$$(x \oplus I) \oplus (y \oplus I) = (x \oplus y) \oplus (I \oplus I) \subseteq (x \oplus y) \oplus I \subseteq U,$$

then (A, τ) is a paratopological MV-algebra. Now, we show that T_a is an open map. Let $a \in A$ and $U \in \tau$. For $y \in a \oplus U$, there exists $x \in U$ such that $y = a \oplus x$. Hence,

$$I \oplus y = I \oplus (a \oplus x) = a \oplus (I \oplus x) \subseteq a \oplus U,$$

and so $a \oplus U$ is an open set. Since for any $a \in A$, T_a is an open map and I is an open set containing 0, (A, τ) is a T_0-space by Proposition 3.1.
(ii) Let (A, τ) be a regular space and $x \in U \in \tau$. Since $I \oplus x$ is an open neighborhood of x, there exists an open set H such that $x \in H \subseteq \overline{H} \subseteq I \oplus x$. Then, $I \oplus x = H = \overline{H}$ and so $I \oplus x$ is a closed set. Conversely, let $x \in U \in \tau$. Since $I \oplus x$ is a closed set, then $x \in I \oplus x = \overline{I \oplus x} \subseteq U$. Hence (A, τ) is a regular space.
(iii) The proof is similar to (ii). □

Theorem 4.4. *Let F be a non-trivial filter of the MV-algebra A. Then there exists a non-trivial topology τ such that the operation $\odot$ is continuous, F is a clopen set and (A, τ) is a*
(i) T_0-space,
(ii) regular space if and only if for any $x \in U \in \tau$, $F \odot x$ is a closed set.
(iii) normal space if and only if for any closed set S, $\bigcup_{x \in S} (F \odot x)$ is a closed set.

Proof. The proof is similar to Theorem 4.3 by putting $\tau = \{U \subseteq A : \forall x \in U,\ F \odot x \subseteq U\}$. □

Example 4.5. *Let A be the MV-algebra in Example 3.6.*
(i) Consider the ideal $I = \{0, a\}$. According to Theorem 4.3, if we put

$$\tau = \{A, \emptyset, \{0, a\}, \{1, b\}, \{a\}, \{1\}, \{a, b, 1\}, \{a, 1, 0\}, \{a, 1\}\},$$

then (A, τ) is a paratopological MV-algebra and T_0-space. But (A, τ) is not a topological MV-algebra by Theorem 3.5, since it is not a Huasdorff space.
(ii) Consider the filter $F = \{1, b\}$. We construct the topology τ as follows:

$$\tau = \{A, \emptyset, \{0, a\}, \{1, b\}, \{b\}, \{0\}, \{a, b, 0\}, \{0, b, 1\}, \{b, 0\}\}.$$

By Theorem 4.4, the operation $\odot$ is continuous and (A, τ) is a T_0-space.

5 Locally compact Hausdorff MV-algebras

In this section, we investigate the conditions under which the locally compact Hausdorff MV-algebras become paratopological MV-algebras and normal spaces.

Proposition 5.1. *Let (A, τ) be a semitopological MV-algebra and locally compact Hausdorff space such that for any $b \in A$, T_b is an open map and the operation $\oplus$ is continuous at $(0, b)$. Then for any $x, y \in A$, there exist open neighborhoods U and V of x and y, respectively, such that $\overline{U \oplus V}$ is compact.*

Proof. Let $x, y \in A$ and $y \in W \in \tau$. Since $0 \oplus y \in W$, there exist open neighborhoods U_0 and V of 0 and y, respectively, such that $U_0 \oplus V \subseteq W$. Put $U = x \oplus U_0$, then U is an open set and contains x. Also, we have

$$U \oplus V = (x \oplus U_0) \oplus V = x \oplus (U_0 \oplus V) \subseteq x \oplus W.$$

Since A is a locally compact Hausdorff space, $\overline{W}$ is compact. Hence $x \oplus \overline{W}$ is a compact and closed set, and so $\overline{x \oplus W} \subseteq \overline{x \oplus \overline{W}} = x \oplus \overline{W}$. On the other hand,

$$x \oplus \overline{W} = T_x(\overline{W}) \subseteq \overline{T_x(W)} = \overline{x \oplus W}.$$

Therefore, $\overline{x \oplus W} = x \oplus \overline{W}$, and so $\overline{x \oplus W}$ is compact. Thus $\overline{U \oplus V}$ is also a compact set, since $\overline{U \oplus V} \subseteq \overline{x \oplus W}$. □

Theorem 5.2. *(A, τ) is a paratopological MV-algebra if the following conditions hold for any $x, y \in A$:*
(i) there are two open sets U_x and V_y of x and y, respectively, such that $\overline{U_x \oplus V_y}$ is compact,
(ii) for any open set W of $x \oplus y$ and any $z \in A \setminus W$, there exist open sets U_x and V_y of x and y, respectively, such that $z \notin \overline{U_x \oplus V_y}$.

Proof. Let $x, y \in A$ and W be an open neighborhood of $x \oplus y$. We must show that there exist open sets U and V of x and y, respectively, such that $U \oplus V \subseteq W$. By (i) there exist open sets U_x and V_y such that $x \in U_x$, $y \in V_y$, and $\overline{U_x \oplus V_y}$ is compact. Let $\mathcal{B}_x$ be the family of all open neighborhoods of x contained in U_x and $\mathcal{B}_y$ be the family of all open neighborhoods of y contained in V_y. For $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$, put $F_{U,V} = (A \setminus W) \cap \overline{U \oplus V}$. Clearly, $F_{U,V}$ is closed. Now, let $\eta = \{F_{U,V} : U \in \mathcal{B}_x, V \in \mathcal{B}_y\}$. Obviously, any element of η is compact. We show that at least one element of η is empty. Assume that all elements of η are non-empty. Since the elements of η are closed and compact sets, by finite intersection property, there exists $z \in \cap\eta$. On the other hand, by (ii) there exist $U \in \mathcal{B}_x$ and $V \in \mathcal{B}_y$ such that $z \notin \overline{U \oplus V}$. Therefore, $z \notin F_{U,V} \in \eta$ which is a contradiction. Then, for some $F_{U,V} \in \eta$, $F_{U,V} = \emptyset$ and so $U \oplus V \subseteq W$. It follows that (A, τ) is a paratopological MV-algebra. □

Example 5.3. *Let A be the MV-algebra in Example 3.6. We consider the following topology on A,*

$$\tau = \{A, \emptyset, \{0, b\}, \{1, a\}, \{b\}, \{1\}, \{a, b, 1\}, \{b, 1, 0\}, \{b, 1\}\}.$$

It is clear that the conditions of Theorem 5.2 hold. Thus (A, τ) is a paratopological MV-algebra.

Proposition 5.4. *Let (A, τ) be a paratopological MV-algebra and locally compact Hausdorff space such that for any $a \in A$, T_a is an open map. Then for any open set U and compact set $C \subseteq U$, there exists an open neighborhood V of 0 such that $\overline{C \oplus V}$ is a compact subset of U.*

Proof. Let U be an open set, C be a compact subset of U, and $x \in C$. Since $\oplus$ is continuous, there exist open neighborhoods V_x and W_x of 0 such that $x \oplus V_x \subseteq U$ and $W_x \oplus W_x \subseteq V_x$. Also, $\{x \oplus W_x : x \in C\}$ is an open cover of C, since for any $a \in A$, T_a is an open map. This cover has a finite subcover such as $\{x_i \oplus W_{x_i} : x_i \in C, i = 1, 2, ..., n\}$. Put $W = \cap_{i=1}^{n} W_{x_i}$, then

$$C \oplus W \subseteq C \oplus W_{x_1} \subseteq (\cup_{i=1}^{n}(x_i \oplus W_{x_i})) \oplus W_{x_1} \subseteq (\cup_{i=1}^{n}(x_i \oplus W_{x_i})) \oplus W_{x_i} \subseteq \cup_{i=1}^{n}(x_i \oplus V_{x_i}) \subseteq U.$$

Since (A, τ) is a locally compact space, there exists an open set V with compact closure containing 0 such that $\overline{V} \subseteq W$. By continuity of T_a, $C \oplus \overline{V}$ is a compact subset of A. Since (A, τ) is a Hausdorff space, $C \oplus \overline{V}$ is a closed subset of A. Thus, $\overline{C \oplus V} \subseteq \overline{C \oplus \overline{V}} = C \oplus \overline{V}$. On the other hand,

$$C \oplus \overline{V} = \cup_{x \in C} T_x(\overline{V}) \subseteq \cup_{x \in C} \overline{T_x(V)} = \overline{C \oplus V}.$$

Hence $\overline{C \oplus V}$ is a compact subset of U. □

Theorem 5.5. *Let (A, τ) be a paratopological MV-algebra and locally compact Hausdorff space. If for any $a \in A$, T_a is an open map, then (A, τ) is a normal space.*

Proof. Let F be a closed set and $F \subseteq V \in \tau$. By Proposition 5.4, there exists an open neighborhood U of 0 such that $F \subseteq U \oplus F \subseteq \overline{U \oplus F} \subseteq V$, which implies that (A, τ) is a normal space. □

The following examples show the necessity of the condition in Theorem 5.5 and indicate that the converse of this theorem is not true in general.

Example 5.6. *Consider the MV-algebra S_2 with topology $\tau = \{S_2, \emptyset, \{\frac{1}{2}\}, \{1, \frac{1}{2}\}, \{1\}\}$. Although for any $a \in S_2$, T_a is an open map and (S_2, τ) is a compact space, it is not a normal space. The reason is that (S_2, τ) is not a paratopological MV-algebra and Hausdorff space.*

Example 5.7. *Let S be the standard MV-algebra with subspace topology τ of $\mathbb{R}$. (S,τ) is a paratopological MV-algebra and normal space. Also, (S,τ) is a compact Hausdorff space. But T_a is not an open map for any $a \in S$ (for example $T_{\frac{1}{2}}(\frac{1}{2},1] = \{1\}$ is not an open set).*

Lemma 5.8. [2] *Let X and Y be locally compact Hausdorff spaces, f be a separately continuous map of $X \times Y$ to a regular space Z and $(x,y) \in X \times Y$. Let W be an open set of $f(x,y)$ and U be an open set of x, then there exist a non-empty open set U_1 in X, and an open set V in Y such that $U_1 \subseteq U$, $y \in V$ and $f(U_1 \times V) \subseteq W$.*

Theorem 5.9. *Let (A,τ) be a locally compact Hausdorff space. If for any $a \in A$, T_a is an open map and $\oplus$ is continuous at $(0,a)$, then (A,τ) is a paratopological MV-algebra and normal space.*

Proof. First, we prove that (A,τ) is a semitopological MV-algebra. Let $x \oplus y \in U \in \tau$. Since $\oplus$ is continuous at $(0, x \oplus y)$, there exists an open neighborhood V of 0 such that $V \oplus (x \oplus y) \subseteq U$. Also, $W = V \oplus x$ is an open neighborhood of x and $W \oplus y \subseteq U$. Hence A is a semitopological MV-algebra. Now, we show that the conditions of Theorem 5.2 hold. Condition (i) holds by Proposition 5.1. Let $x \oplus y \in W \in \tau$ and $z \in A \setminus W$. Since A is locally compact, we can assume $z \notin \overline{W}$. Since $0 \oplus z = z \in A \setminus \overline{W}$ and $\oplus$ is continuous at $(0,z)$, there exists an open neighborhood G of 0 such that $(G \oplus z) \cap \overline{W} = \emptyset$. Also, $G \oplus x$ is an open neighborhood of x. Then by Lemma 5.8, there exist two non-empty open sets U_0 and V such that $y \in V$, $U_0 \subseteq G \oplus x$ and $U_0 \oplus V \subseteq W$. Thus, there exists $g \in G$ such that $g \oplus x \in U_0$. By continuity of T_a, there exists $U \in \tau$ containing 0 such that $g \oplus U \subseteq U_0$. Therefore,

$$(g \oplus U) \oplus V \subseteq U_0 \oplus V \subseteq W \Longrightarrow g \oplus \overline{U \oplus V} = T_g(\overline{U \oplus V}) \subseteq \overline{g \oplus (U \oplus V)} \subseteq \overline{W}.$$

We claim $z \notin \overline{U \oplus V}$. Let $z \in \overline{U \oplus V}$, then

$$g \oplus z \in g \oplus \overline{U \oplus V} \Longrightarrow g \oplus z \in \overline{W} \cap (G \oplus z),$$

which is a contradiction. Hence condition (ii) also holds. Therefore, (A,τ) is a paratopological MV-algebra and consequently, it is a normal space by Theorem 5.5. □

6 Hausdorff topological quotient MV-algebras

Let I be an ideal of MV-algebra A. Let $\frac{A}{I}$ be a quotient MV-algebra and $\pi_I : A \longrightarrow \frac{A}{I}$ be the natural homomorphism. A topology is defined on $\frac{A}{I}$ as follows: A subset U of $\frac{A}{I}$ is open if $\pi_I^{-1}(U)$ is an open subset of A. This topology on $\frac{A}{I}$ denoted by $\widetilde{\tau}$ is called the quotient topology induced by π_I. It is well known that $\widetilde{\tau}$ is the largest topology on $\frac{A}{I}$ making π_I continuous. In this section, the quotient MV-algebra $\frac{A}{I}$ equipped with the topology $\widetilde{\tau}$ are studied.

Lemma 6.1. *Let I be an ideal of MV-algebra A. Let τ be a topology on A and $\widetilde{\tau}$ be the quotient topology on $\frac{A}{I}$.*
(i) If $V \in \widetilde{\tau}$, then there exists $U \in \tau$ such that $\pi_I(U) = V$.
(ii) For each $x \in A$, $\left(\pi_I^{-1} \circ \pi_I\right)\left(\frac{x}{I}\right) = \frac{x}{I}$. Also $\left(\pi_I^{-1} \circ \pi_I\right)(S) = \bigcup_{x \in S} \frac{x}{I}$, for each $S \subseteq A$.
(iii) If π_I is an open set and (A,τ) is a topological MV-algebra, then $(\frac{A}{I}, \widetilde{\tau})$ is also a topological MV-algebra.

Proof. It is straightforward by definitions. □

Proposition 6.2. *Let (A,τ) be a topological MV-algebra and I be an ideal of A.*
(i) 0 is an interior point of I if and only if for each $x\in A$, $\frac{x}{I}$ is an open subset of A.
(ii) I is a closed subset of A if and only if for each $x\in A$, $\frac{x}{I}$ is a closed subset of A.

Proof. (i) Let 0 be an interior point of I. By Proposition 2.13, I is an open set of A. Since (A,τ) is a topological MV-algebra, L_x and R_x are continuous for any $x\in A$, and so $\frac{x}{I}=L_x^{-1}(I)\cap R_x^{-1}(I)$ is an open set. Conversely, if for any $x\in A$, $\frac{x}{I}$ is an open set, then $\frac{0}{I}=I$ is an open set and so 0 is an interior point of I.
(ii) The proof is similar to (i). □

Definition 6.3. [5] *We say that (A,τ) satisfies the open condition if for any ideal I, π_I is an open map.*

In the following example, we show that a natural homomorphism may not be an open map.

Example 6.4. *Let A be the MV-algebra in Example 3.6.*
(i) If $\tau=\{A,\emptyset,\{0,a\},\{1,b\}\}$, then (A,τ) satisfies the open condition.
(ii) $\tau=\{A,\emptyset,\{0\},\{a,b,1\}\}$ is a topology on A. If $I=\{0,a\}$, then $\pi_I(\{0\})=\{0,a\}$. Hence π_I is not an open map.

Proposition 6.5. *Let (A,τ) be a topological MV-algebra and I be an ideal of A.*
(i) If $\pi_I(0)$ is an open subset of $\frac{A}{I}$, then π_I is a closed map.
(ii) If 0 is an interior point of I, then π_I is an open map.
(iii) If I is a closed subset of A and $\frac{A}{I}$ is a finite quotient MV-algebra, then π_I is a closed map.

Proof. (i) Let $\pi_I(0)$ be an open subset of $\frac{A}{I}$ and U be a closed subset of A. We show that $\pi_I(U)$ is also a closed set. Since π_I is a continuous map, $\pi_I(U)=\pi_I(\overline{U})\subseteq\overline{\pi_I(U)}$. Now, let $\frac{y}{I}\in\overline{\pi_I(U)}$. Then there exists a net $\{x_j:j\in J\}\subseteq U$ such that $\{\frac{x_j}{I}:j\in J\}$ converges to $\frac{y}{I}$. By continuity of $\ominus$, the nets $\left\{\frac{x_j\ominus y}{I}:j\in J\right\}$ and $\left\{\frac{y\ominus x_j}{I}:j\in J\right\}$ converge to $\frac{0}{I}$. Then there exists $j\in J$ such that $\frac{x_j\ominus y}{I}$ and $\frac{y\ominus x_j}{I}$ belong to $\pi_I(0)$. Consequently, $\frac{x_j\ominus y}{I}=\frac{0}{I}=\frac{y\ominus x_j}{I}$. Hence $\frac{x_j}{I}=\frac{y}{I}$ which implies that $\frac{y}{I}\in\pi_I(U)$ and so $\overline{\pi_I(U)}\subseteq\pi_I(U)$. Therefore, $\pi_I(U)=\overline{\pi_I(U)}$.
(ii) Let 0 be an interior point of I. By Lemma 6.1 and Proposition 6.5, $\left(\pi_I^{-1}\circ\pi_I\right)(U)=\bigcup_{x\in U}\frac{x}{I}$ is an open set of A which implies $\pi_I(U)$ is an open set in $\frac{A}{I}$.
(iii) Let I be a closed set and U be a closed subset of A. Since $\frac{A}{I}$ is finite, there are $x_1,\dots,x_n\in A$ such that $(\pi_I^{-1}\circ\pi_I)(U)=\bigcup_{j=1}^{n}\frac{x_j}{I}$, by Lemma 6.1. Also, by Proposition 6.5, $(\pi_I^{-1}\circ\pi_I)(U)$ is a closed subset of A which implies $\pi_I(U)$ is closed. □

Theorem 6.6. *Let (A,τ) be a topological MV-algebra and I be an ideal of A.*
(i) If 0 is an interior point of I, then $(\frac{A}{I},\widetilde{\tau})$ is a Hausdorff topological quotient MV-algebra.
(ii) I is a closed subset of A if and only if $(\frac{A}{I},\widetilde{\tau})$ is a T_1-space.

Proof. (i) Let 0 be an interior point of I. By Proposition 6.2, for any $x\in A$, $\frac{x}{I}$ is an open subset of A. Then for any $U\in\tau$, $(\pi_I^{-1}\circ\pi_I)(U)=\bigcup_{x\in U}\frac{x}{I}$ is an open subset of A. Also, by Proposition 6.5, π_I is an open map. Therefore, by Lemma 6.1, $(\frac{A}{I},\widetilde{\tau})$ is a topological quotient MV-algebra. Also, for any $x\in A$, $\pi_I(\frac{x}{I})=\{\frac{x}{I}\}$ is an open subset of $\frac{A}{I}$. Hence $\frac{A}{I}$ is a discrete space and so it is a Hausdorff space.
(ii) Let I be a closed subset of A, then by Proposition 6.1, for any $x\in A$, $(\pi_I^{-1}\circ\pi_I)(\frac{x}{I})=\frac{x}{I}$ is a closed subset of A. Therefore, $\pi_I(\frac{x}{I})=\{\frac{x}{I}\}$ is closed in $\frac{A}{I}$ which implies that $\frac{A}{I}$ is a T_1-space. Conversely, if $\frac{A}{I}$ is a T_1-space, then $\pi_I^{-1}(\{\frac{0}{I}\})=I$ is a closed subset of A. □

Example 6.7. *Let (A, τ) be the topological MV-algebra in Example 3.6(i).*
(i) *If $I = \{0, a\}$, then $(\frac{A}{I}, \widetilde{\tau})$ is a Hausdorff topological quotient MV-algebra by Theorem 6.6.*
(ii) *If $I = \{0, b\}$, then $(\frac{A}{I}, \widetilde{\tau})$ is not a T_1-space by Theorem 6.6.*

Proposition 6.8. *Let (A, τ) be a topological MV-algebra and I be an ideal of A. Suppose $\mathcal{W}$ is a fundamental system of neighborhoods of 0. If 0 is an interior point of I, then $\bigcap_{W \in \mathcal{W}} \pi_I(W) = \{\frac{0}{I}\}$ and $\bigcap_{W \in \mathcal{W}} W \subseteq I$.*

Proof. Let 0 be an interior point of I. Then $(\frac{A}{I}, \widetilde{\tau})$ is a Hausdorff topological quotient MV-algebra by Theorem 6.6(i). Also, π_I is an open map by Theorem 6.5(ii). Hence $\{\pi_I(W) : W \in \mathcal{W}\}$ is a fundamental system of neighborhoods of $\frac{0}{I}$. Let $\frac{0}{I} \neq \frac{x}{I} \in \bigcap_{W \in \mathcal{W}} \pi_I(W)$. Since $(\frac{A}{I}, \widetilde{\tau})$ is a Hausdorff topological quotient MV-algebra, then by Proposition 2.12, $\frac{1}{I^*} \neq \frac{x^*}{I^*} \in \bigcap_{W^* \in \mathcal{W}^*} \pi_{I^*}(W^*)$ which is a contradiction by Theorem 3.3. Therefore, $\bigcap_{W \in \mathcal{W}} \pi_I(W) = \{\frac{0}{I}\}$. Now, Let $x \in \bigcap_{W \in \mathcal{W}} W$. Then $\pi_I(x) \in \bigcap_{W \in \mathcal{W}} \pi_I(W) = \{\frac{0}{I}\}$. Hence $\frac{x}{I} = \frac{0}{I}$ which implies that $x \in I$ and so $\bigcap_{W \in \mathcal{W}} W \subseteq I$. □

Theorem 6.9. *Let (A, τ) be a topological MV-algebra and I be an ideal of A.*
(i) *If $\{0\}$ is an open subset of A, then $(\frac{A}{I}, \widetilde{\tau})$ is a Hausdorff topological quotient MV-algebra.*
(ii) *If $\{1\}$ is a closed subset of A, then $(\frac{A}{I}, \widetilde{\tau})$ is a T_1-space.*

Proof. (i) Let $\{0\}$ be an open subset of A. By Theorem 3.7(ii), (A, τ) is a discrete space. Since π_I is a continuous map, $(\frac{A}{I}, \widetilde{\tau})$ is a discrete space and so it is a Hausdorff topological quotient MV-algebra.
(ii) Let $\{1\}$ be a closed subset of A. By Theorem 3.7(i), (A, τ) is a T_1-space. Since π_I is a continuous and surjective map, $(\frac{A}{I}, \widetilde{\tau})$ is also a T_1-space. □

Proposition 6.10. *Let (A, τ) be a topological MV-algebra which satisfies the open condition and I be an ideal of A.*
(i) *$(\frac{A}{I}, \widetilde{\tau})$ is a Hausdorff topological quotient MV-algebra.*
(ii) *If I is an maximal ideal of A, then I is dense in A or $(\frac{A}{I}, \widetilde{\tau})$ is a Hausdorff topological quotient MV-algebra.*

Proof. (i) Since (A, τ) is a topological MV-algebra, $\overline{I}$ is an ideal of A by Proposition 3.9. Then $(\frac{A}{\overline{I}}, \widetilde{\tau})$ is a T_1-space by Theorem 6.6(ii). Since $\pi_{\overline{I}}$ is an open map, $(\frac{A}{\overline{I}}, \widetilde{\tau})$ is a topological quotient MV-algebra. Therefore, (A, τ) is a Hausdorff space by Theorem 3.5.
(ii) Let I be a maximal ideal of A, then $\overline{I} = A$ or $\overline{I} = I$. If $\overline{I} = A$, then I is dense in A. If $\overline{I} = I$, then $(\frac{A}{I}, \widetilde{\tau})$ is a Hausdorff topological quotient MV-algebra by (i). □

Conclusion

In this paper, we discussed the relations between T_i-spaces $(i = 0, 1, 2)$ in (quasi)topological MV-algebras. We applied prefilters, filters and ideals to construct some topologies on MV-algebras. The conditions under which locally compact Hausdorff MV-algebras can be paratopological MV-algebras and normal spaces were derived. Also, we explored the conditions for a quotient MV-algebra to be a Hausdorff topological quotient MV-algebra. By using the ideas and results obtained in this paper, the other concepts of topology can be studied on (quasi, para)topological MV-algebras. Moreover, the obtained results can be applied to other algebraic structures in the future.

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