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Block code on L-algebras

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Abstract

By using the notion of L-algebras as an important part of the ordered algebra, we introduce the notions of block code, x-function and x-subsets on an arbitrary L-algebra. Then some related properties and examples are provided. Also, by using these notions, we define an equivalence relation on L-algebra and we introduce a new order on the generated code based on L-algebras. Finally, we will provide a method which allows us to find an L-algebra starting from a given arbitrary binary block code.

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1 Introduction

The quantum Yang-Baxter equation (QYBE for short) was created by Zhenning Yang and R. J. Baxter in 1967 and 1972, respectively, which is the most fundamental equation in the field of mathematical physics [8]. QYBE is closely related to a series of mathematical structures, such as quantum binomial algebras [10, 11], I-type semigroups and Bieberbach groups [12, 18], plane curves and dyeing of bijective 1-type cocycles [8], semimunipolar small triangular Hopf algebra [20], dynamic system [6], geometric crystal [5], etc. In 2005, W. Rump studied the algebraic solution of the quantum Yang-Baxter equation.

For any set X, a binary operation $\bullet$ is defined that applies to the following equation (L)

$$(x \bullet y) \bullet (x \bullet z) = (y \bullet x) \bullet (y \bullet z), \qquad (L)$$

He called this structure $(X, \bullet)$ as an L-algebra. If the left-multiplication mapping $x \mapsto y \bullet x$ is bijective, the structure $(X, \bullet)$ is said to be a self-similar L-algebra. The authors proved that

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the self-similar L-algebra corresponds to a solution of the quantum Yang-Baxter equation. Then, Rump started from a self-similar L-algebra and naturally induced a new binary operation $*$, called a product, which makes $(X, \bullet, *)$ a left hoop. Then, the author studied the group structure on self-similar L-algebras, and discussed the relationship between such group structure and L-groups. These results play an important role in the study of the solution of the quantum Yang-Baxter equation. On the other hand, the equation (L) also appears in some algebraic logic systems. If the elements of the set are treated as propositions and $x \bullet y$ as implication operation $x \rightarrowtail y$, the equation (L) becomes a truth-degree description of classical logic, intuitionist logic, or Łukasiewicz infinite value propositional logic. Recently, the equation (L) has been applied to quantum logic and its algebraic models. (see [3, 19] and [14]).

In coding theory, a block code is an error-correcting code which encode data in blocks. Recently, many mathematicians have studied codes on different logical algebras and have obtained many results in this field. For instance, Jahanshahi investigated binary block code on MV-algebras and Flaut investigated this notion on BCK-algebras and Surdive investigated this notion on hyper BCK-algebars and etc. For more details you can see [1, 2, 9, 13]. Due to the importance of this matter, we decided to examine this issue on L-algebras.

In this paper, by using the notion of L-algebras, we introduce the notions of block code, x-function and x-subsets on an arbitrary L-algebra. Then, we define an equivalence relation on L-algebra and we introduce a new order on the generated code based on L-algebras. We have to notice that the main motivation of this work is that by using an algebraic structure that is a lattice, it is possible to create block codes, and vice versa, by using arbitrary given binary codes and using the lexicographic relation to create an L-algebra, that is, a binary operation of the definition to satisfy the characteristics of L-algebra.

2 Preliminaries

This section lists the known default contents that will be used later.
An L-algebra [5] is an algebraic structure $(\mathcal{L}; \rightarrowtail, 1)$ of type $(2, 1)$ satisfying
$(L1)$ $x \rightarrowtail x = x \rightarrowtail 1 = 1$, and $1 \rightarrowtail x = x$,
$(L2)$ $(x \rightarrowtail y) \rightarrowtail (x \rightarrowtail z) = (y \rightarrowtail x) \rightarrowtail (y \rightarrowtail z)$,
$(L3)$ if $x \rightarrowtail y = y \rightarrowtail x = 1$, then $x = y$,
for any $x, y, z \in \mathcal{L}$. Condition $(L1)$ states that 1 is a logical unit, while $(L2)$ is related to the quantum Yang-Baxter equation. Note that a logical unit is always unique.

By $(L3)$, the relation

$$x \leq y \text{ if and only if } x \rightarrowtail y = 1,$$

defines a partial order for any L-algebra $\mathcal{L}$. If $\mathcal{L}$ admits a smallest element 0, we speak of a bounded L-algebra.

Definition 2.1. [15] *An L-algebra $(\mathcal{L}, \rightarrowtail, 1)$ which satisfies in the following condition*

$$x \rightarrowtail (y \rightarrowtail x) = 1, \qquad (K)$$

*for any $x, y \in \mathcal{L}$ is called a KL-*algebra.

Proposition 2.2. [17] *Let $(\mathcal{L}; \rightarrowtail, 1)$ be an L-algebra. Then $x \leq y$ implies $z \rightarrowtail x \leq z \rightarrowtail y$, for any $x, y, z \in \mathcal{L}$.*

Proposition 2.3. [17] *For an L-algebra $(\mathcal{L}; \rightarrowtail, 1)$, the following are equivalent:*
(i) $x \leq y \rightarrowtail x$,

(ii) *if* $x \leq z$, *then* $z \rightarrowtail y \leq x \rightarrowtail y$,
(iii) $((x \rightarrowtail y) \rightarrowtail z) \rightarrowtail z \leq ((x \rightarrowtail y) \rightarrowtail z) \rightarrowtail ((y \rightarrowtail x) \rightarrowtail z)$, *for any* $x, y, z \in \mathcal{L}$.

Note. From now on, in this paper $(\mathcal{L}; \rightarrowtail, 1)$ or $\mathcal{L}$, for short, is an L-algebra, unless otherwise state.

3 x-function and x-subset on L-algebras

In this section, we introduce the notations of x-function and x-subset on L-algebras and we investigate some properties of them.

Definition 3.1. *Assume* $\mathcal{A}$ *is a non-empty set. For any* L*-algebra* $\mathcal{L}$*, a mapping* $\mathfrak{f} : \mathcal{A} \to \mathcal{L}$ *is said to be an* ℓ*-function on* $\mathcal{A}$ *based on* $\mathcal{L}$ *and is denoted by* $\mathfrak{f}_{\mathcal{L}}$*. If there is no confusion about* $\mathcal{L}$*, we use* $\mathfrak{f}$ *instead of* $\mathfrak{f}_{\mathcal{L}}$*.*

The set of all ℓ*-functions on* $\mathcal{A}$ *based on* $\mathcal{L}$ *is denoted by* $\mathcal{LF}(\mathcal{A})$*.*

Definition 3.2. *Consider* $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$ *and* $x \in \mathcal{L}$*. Define the function* $\widehat{\mathfrak{f}}_x : \mathcal{A} \to \{0, 1\}$*, where for any* $a \in \mathcal{A}$*, we have*

$$\widehat{\mathfrak{f}}_x(a) = \begin{cases} 1 & \mathfrak{f}(a) \rightarrowtail x = 1 \\ 0 & o.w \end{cases}$$

Then the function $\widehat{\mathfrak{f}}_x$ *is called an* x*-function of* $\mathcal{L}$*. The set of all* x*-functions on* $\mathcal{L}$ *is denoted by* Φ*.*

In addition, define $\mathcal{A}_x$ *for any* $x \in \mathcal{L}$ *as follows:*

$$\mathcal{A}_x = \{a \in \mathcal{A} \mid \mathfrak{f}(a) \rightarrowtail x = 1\} = \{a \in \mathcal{A} \mid \mathfrak{f}(a) \leq x\}, \tag{1}$$

which is called an x*-subset of* $\mathcal{A}$*. The set of all* x*-subsets of* $\mathcal{A}$ *is denoted by* $\mathcal{S}$*.*

Remark 3.3. (1) *Clearly, by (L1),* $\mathcal{A}_1 = \{a \in \mathcal{A} \mid \mathfrak{f}(a) \rightarrowtail 1 = 1\} = \mathcal{A}$*, and if* $\mathcal{L}$ *is a bounded* L*-algebra, then*

$$\mathcal{A}_0 = \{a \in \mathcal{A} \mid \mathfrak{f}(a) \rightarrowtail 0 = 1\} = \{a \in \mathcal{A} \mid \mathfrak{f}(a) = 0\} = \mathfrak{f}^{-1}(0).$$

(2) *Since for* $a \in \mathcal{A}$*, by (L1),* $\mathfrak{f}(a) \rightarrowtail \mathfrak{f}(a) = 1$*, we get* $a \in \mathcal{A}_{\mathfrak{f}(a)}$*.*

Example 3.4. *Let* $\mathcal{A} = \{0, x, y, z\}$ *be a set and* $(\mathcal{L} = \{0, a, b, 1\}, \leq)$ *be a chain where* $0 \leq a \leq b \leq 1$*. Define the operation* $\rightarrowtail$ *on* $\mathcal{L}$ *as follows:*

$\rightarrowtail$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

Then $(\mathcal{L}, \rightarrowtail, 0, 1)$ *is a bounded* L*-algebra. Define* $\mathfrak{f} : \mathcal{A} \to \mathcal{L}$ *such that*

$$\mathfrak{f}(0) = 0,\ \mathfrak{f}(x) = a,\ \mathfrak{f}(y) = b \text{ and } \mathfrak{f}(z) = 1.$$

Then $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$*. Also, we have*

$$\mathcal{A}_0 = \{d \in \mathcal{A} \mid \mathfrak{f}(d) \rightarrowtail 0 = 1\} = \{0\}, \qquad \mathcal{A}_a = \{d \in \mathcal{A} \mid \mathfrak{f}(d) \rightarrowtail a = 1\} = \{0, x\},$$
$$\mathcal{A}_b = \{d \in \mathcal{A} \mid \mathfrak{f}(d) \rightarrowtail b = 1\} = \{0, x, y\}, \qquad \mathcal{A}_1 = \mathcal{A}.$$

Proposition 3.5. *Consider $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$. Then for $a \in \mathcal{A}$, $\mathfrak{f}$ can be represented by the infimum of the set $\{x \in \mathcal{L} \mid \hat{\mathfrak{f}}_x(a) = 1\}$ regarding the partial order $\leq$. It means that*

$$\forall\ a \in \mathcal{A}, \qquad \mathfrak{f}(a) = \inf\{x \in \mathcal{L} \mid \hat{\mathfrak{f}}_x(a) = 1\}.$$

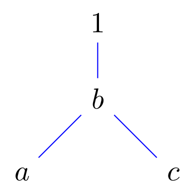
Proof. Let $\mathcal{B} = \{x \in \mathcal{L} \mid \hat{\mathfrak{f}}_x(a) = 1\}$ and $a \in \mathcal{A}$. Since $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$, we get $\mathfrak{f}(a) \in \mathcal{L}$, so there exists $x \in \mathcal{L}$ such that $\mathfrak{f}(a) = x$ and so $\mathfrak{f}(a) \rightarrowtail x = 1$. Thus $a \in \mathcal{A}_x$ and $\hat{\mathfrak{f}}_x(a) = 1$. Hence, $x \in \mathcal{B}$. Now, if $z \in \mathcal{L}$ such that $\hat{\mathfrak{f}}_z(a) = 1$, then $\mathfrak{f}(a) \rightarrowtail z = 1$. Since $\mathfrak{f}(a) = x$, we get $x \rightarrowtail z = 1$. Hence, $x = \inf \mathcal{B}$. Therefore, $\mathfrak{f}(a) = \inf\{x \in \mathcal{L} \mid \hat{\mathfrak{f}}_x(a) = 1\}$. □

Proposition 3.6. *Let $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$, $x \leq y$ which $x, y \in \mathcal{L}$. Then $\mathcal{A}_x \subseteq \mathcal{A}_y$.*

Proof. Assume $x, y \in \mathcal{L}$ such that $x \leq y$ and $a \in \mathcal{A}_x$. Then $\mathfrak{f}(a) \rightarrowtail x = 1$, and so by Proposition 2.2, $1 = \mathfrak{f}(a) \rightarrowtail x \leq \mathfrak{f}(a) \rightarrowtail y$. Thus, $\mathfrak{f}(a) \rightarrowtail y = 1$. Hence, $a \in \mathcal{A}_y$. Therefore, $\mathcal{A}_x \subseteq \mathcal{A}_y$. □

By the following example we show that the converse of Proposition 3.6 does not hold:

Example 3.7. *Let $\mathcal{A} = \{x, y, z, w\}$ be a set and $(\mathcal{L} = \{a, b, c, 1\}, \leq)$ be a poset with the following Hasse diagram:*



Define the operation $\rightarrowtail$ on $\mathcal{L}$ as follows:

$\rightarrowtail$	a	b	c	1
a	1	1	a	1
b	a	1	c	1
c	a	1	1	1
1	a	b	c	1

Then $(\mathcal{L}, \rightarrowtail, 1)$ is an L-algebra. Define $\mathfrak{f} : \mathcal{A} \to \mathcal{L}$ by:

$$\mathfrak{f}(x) = a,\ \mathfrak{f}(y) = a,\ \mathfrak{f}(z) = b,\ \mathfrak{f}(w) = 1.$$

Then $\mathcal{A}_a = \{x, y\}$, $\mathcal{A}_c = \emptyset$, $\mathcal{A}_b = \{x, y, z\}$ and $\mathcal{A}_1 = \mathcal{A}$. Clearly, $\mathcal{A}_c \subseteq \mathcal{A}_a$, but a and c are incomparable.

Theorem 3.8. *Consider $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$. Then, we have*
(i) for any $a_1, a_2 \in \mathcal{A}$, $\mathfrak{f}(a_1) \neq \mathfrak{f}(a_2)$ if and only if $\mathcal{A}_{\mathfrak{f}(a_1)} \neq \mathcal{A}_{\mathfrak{f}(a_2)}$.
(ii) for any $x \in \mathcal{L}$ and for any $a \in \mathcal{A}$, $\mathfrak{f}(a) \leq x$ if and only if $\mathcal{A}_{\mathfrak{f}(a)} \subseteq \mathcal{A}_x$.

Proof. (i) Suppose $a_1, a_2 \in \mathcal{A}$ such that $\mathcal{A}_{\mathfrak{f}(a_1)} = \mathcal{A}_{\mathfrak{f}(a_2)}$. Since by $(L1)$, $\mathfrak{f}(a_1) \rightarrowtail \mathfrak{f}(a_1) = 1$, we get $a_1 \in \mathcal{A}_{\mathfrak{f}(a_1)} = \mathcal{A}_{\mathfrak{f}(a_2)}$. Then $a_1 \in \mathcal{A}_{\mathfrak{f}(a_2)}$ and so $\mathfrak{f}(a_1) \rightarrowtail \mathfrak{f}(a_2) = 1$. By the similar way, we get $\mathfrak{f}(a_2) \rightarrowtail \mathfrak{f}(a_1) = 1$. Hence, $\mathfrak{f}(a_1) = \mathfrak{f}(a_2)$, which is a contradiction.
Conversely, assume $\mathfrak{f}(a_1) = \mathfrak{f}(a_2)$. Then $\mathfrak{f}(a_1) \rightarrowtail \mathfrak{f}(a_2) = 1$ and $\mathfrak{f}(a_2) \rightarrowtail \mathfrak{f}(a_1) = 1$. Thus by Proposition 3.6, $\mathcal{A}_{\mathfrak{f}(a_1)} \subseteq \mathcal{A}_{\mathfrak{f}(a_2)}$ and $\mathcal{A}_{\mathfrak{f}(a_2)} \subseteq \mathcal{A}_{\mathfrak{f}(a_1)}$, and so $\mathcal{A}_{\mathfrak{f}(a_1)} = \mathcal{A}_{\mathfrak{f}(a_2)}$, which is a contradiction. Hence, $\mathfrak{f}(a_1) \neq \mathfrak{f}(a_2)$.
(ii) Consider for $a \in \mathcal{A}$ and $x \in \mathcal{L}$, $\mathfrak{f}(a) \rightarrowtail x = 1$ and $y \in \mathcal{A}_{\mathfrak{f}(a)}$. Then by (1), $\mathfrak{f}(y) \rightarrowtail \mathfrak{f}(a) = 1$. By assumption, $\mathfrak{f}(y) \rightarrowtail x = 1$ and so $y \in \mathcal{A}_x$. Conversely, clearly, $a \in \mathcal{A}_{\mathfrak{f}(a)}$. By hypothesis, $a \in \mathcal{A}_x$ and so by (1), $\mathfrak{f}(a) \rightarrowtail x = 1$. Hence, $\mathfrak{f}(a) \leq x$. □

Proposition 3.9. *Suppose* $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$ *and* $X \subseteq \mathcal{L}$. *If* $\mu = \inf\{x \mid x \in X\}$ *regarding the partial ordering* $\leq$, *then* $\mathcal{A}_\mu = \bigcap\{\mathcal{A}_x \mid x \in X\}$.

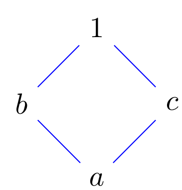
Proof. Assume $a \in \mathcal{A}_\mu$. Then by (1), $\mathfrak{f}(a) \rightarrowtail \mu = 1$. By assumption, $\mu = \inf\{x \mid x \in X\}$. Thus $\mathfrak{f}(a) \rightarrowtail \inf\{x \mid x \in X\} = 1$, and so for any $x \in X$, $\mathfrak{f}(a) \rightarrowtail x = 1$. Hence, for any $x \in X$, $a \in \mathcal{A}_x$. So, $a \in \bigcap\{\mathcal{A}_x \mid x \in X\}$. Hence, $\mathcal{A}_\mu \subseteq \bigcap\{\mathcal{A}_x \mid x \in X\}$. Conversely, let $a \in \bigcap\{\mathcal{A}_x \mid x \in X\}$. Then for any $x \in X$, $a \in \mathcal{A}_x$ and so by (1), $\mathfrak{f}(a) \rightarrowtail x = 1$. Thus $\mathfrak{f}(a) \rightarrowtail \inf\{x \mid x \in X\} = \mathfrak{f}(a) \rightarrowtail \mu = 1$. Hence, $a \in \mathcal{A}_\mu$, and so $\bigcap\{\mathcal{A}_x \mid x \in X\} \subseteq \mathcal{A}_\mu$. Therefore, $\mathcal{A}_\mu = \bigcap\{\mathcal{A}_x \mid x \in X\}$. □

Corollary 3.10. *Consider* $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$ *and* $\mathcal{L}$ *is a* KL*-algebra. Then for any* $x, y \in \mathcal{L}$, $\mathcal{A}_x \subseteq \mathcal{A}_{y \rightarrowtail x}$.

Proof. Let $a \in \mathcal{A}_x$. Then by (1), $\mathfrak{f}(a) \rightarrowtail x = 1$. By (K), we get $x \leq y \rightarrowtail x$, for any $y \in \mathcal{L}$. Thus, by Proposition 2.2, $1 = \mathfrak{f}(a) \rightarrowtail x \leq \mathfrak{f}(a) \rightarrowtail (y \rightarrowtail x)$, and so $\mathfrak{f}(a) \rightarrowtail (y \rightarrowtail x) = 1$, thus, $a \in \mathcal{A}_{y \rightarrowtail x}$. Hence, $\mathcal{A}_x \subseteq \mathcal{A}_{y \rightarrowtail x}$. □

The following example shows that the reverse inclusion in Corollary 3.10 does not hold, in general.

Example 3.11. *Let* $\mathcal{A} = \{x, y, z, w\}$ *be a set and* $(\mathcal{L} = \{a, b, c, 1\}, \leq)$ *be a poset with the following Hasse diagram:*



Define the binary operation $\rightarrowtail$ *on* $\mathcal{L}$ *as follows:*

$\rightarrowtail$	a	b	c	1
a	1	1	1	1
b	c	1	c	1
c	b	b	1	1
1	a	b	c	1

Then $(\mathcal{L}, \rightarrowtail, 1)$ *is an* L*-algebra. Consider* $\mathfrak{f} : \mathcal{A} \to \mathcal{L}$ *by*

$$\mathfrak{f}(x) = a,\ \mathfrak{f}(y) = b,\ \mathfrak{f}(z) = c,\ \mathfrak{f}(w) = 1.$$

Then $\mathcal{A}_a = \{x\}$, $\mathcal{A}_b = \{x, y\}$, $\mathcal{A}_c = \{x, z\}$ *and* $\mathcal{A}_1 = \mathcal{A}$. *Since* $b \rightarrowtail a = c$, *clearly*

$$\mathcal{A}_{b \rightarrowtail a} = \mathcal{A}_c = \{x, z\} \nsubseteq \{x\} = \mathcal{A}_a.$$

Corollary 3.12. *If* $\mathfrak{f} \in \mathcal{FL}(\mathcal{A})$, *then* $\mathcal{A} = \bigcup\{\mathcal{A}_x \mid x \in \mathcal{L}\}$.

Proof. Consider $a \in \mathcal{A}$. Since $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$, $\mathfrak{f}(a) \rightarrowtail \mathfrak{f}(a) = 1$ and $\mathfrak{f}(a) \in \mathcal{L}$, we have $a \in \mathcal{A}_{\mathfrak{f}(a)}$. In addition, clearly, $\mathcal{A}_{\mathfrak{f}(a)} \subseteq \bigcup\{\mathcal{A}_x \mid x \in \mathcal{L}\}$. Thus, $\mathcal{A} \subseteq \bigcup\{\mathcal{A}_x \mid x \in \mathcal{L}\}$. By Remark 3.3, $\mathcal{A}_1 = \mathcal{A}$. So, since $\mathcal{A}_x \subseteq \mathcal{A}$, for any $x \in \mathcal{L}$, we have $\bigcup\{\mathcal{A}_x \mid x \in \mathcal{L}\} \subseteq \mathcal{A}$. Therefore, $\mathcal{A} = \bigcup\{\mathcal{A}_x \mid x \in \mathcal{L}\}$. □

Suppose $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$. Define a binary relation $\sim$ on $\mathcal{L}$ as follows:

$$\forall\, x, y \in \mathcal{L}, \quad x \sim y \ \Leftrightarrow\ \mathcal{A}_x = \mathcal{A}_y. \tag{2}$$

Clearly, $\sim$ is an equivalence relation on $\mathcal{L}$.

Proposition 3.13. *Consider* $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$. *Then for any* $x, y \in \mathcal{L}$

$$x \sim y \;\Leftrightarrow\; \mathfrak{f}(\mathcal{A}) \cap \downarrow x = \mathfrak{f}(\mathcal{A}) \cap \downarrow y,$$

where $\mathfrak{f}(\mathcal{A}) = \{\mathfrak{f}(x) \mid x \in \mathcal{A}\}$ *and* $\downarrow x = \{z \in \mathcal{L} \mid z \leq x\}$.

Proof. Consider $x, y \in \mathcal{L}$ such that $x \sim y$. Then by (2), $\mathcal{A}_x = \mathcal{A}_y$. Suppose $a \in \mathfrak{f}(\mathcal{A}) \cap \downarrow x$. Then there exists $b \in \mathcal{A}$ such that $a = \mathfrak{f}(b)$ and $a \leq x$. Thus, $1 \leq \mathfrak{f}(b) \rightarrowtail x$, and so $\mathfrak{f}(b) \rightarrowtail x = 1$ and by (1), we get $b \in \mathcal{A}_x$. By assumption, since $x \sim y$, we have $\mathcal{A}_x = \mathcal{A}_y$, and so $b \in \mathcal{A}_y$. Thus $\mathfrak{f}(b) \rightarrowtail y = 1$ and so $a = \mathfrak{f}(b) \leq y$. Hence, $a \in \mathfrak{f}(\mathcal{A}) \cap \downarrow y$. The proof of other side is similar. Therefore, $\mathfrak{f}(\mathcal{A}) \cap \downarrow x = \mathfrak{f}(\mathcal{A}) \cap \downarrow y$.

Conversely, suppose $\mathfrak{f}(\mathcal{A}) \cap \downarrow x = \mathfrak{f}(\mathcal{A}) \cap \downarrow y$. Assume $a \in \mathcal{A}_x$. Then by (1), $\mathfrak{f}(a) \rightarrowtail x = 1$ and so $\mathfrak{f}(a) \in \mathfrak{f}(\mathcal{A}) \cap \downarrow x$. By hypothesis, $\mathfrak{f}(a) \in \mathfrak{f}(\mathcal{A}) \cap \downarrow y$. So, $\mathfrak{f}(a) \rightarrowtail y = 1$, and so $a \in \mathcal{A}_y$. Hence, $\mathcal{A}_x \subseteq \mathcal{A}_y$. Similarly, we can prove $\mathcal{A}_y \subseteq \mathcal{A}_x$. Therefore, $\mathcal{A}_x = \mathcal{A}_y$. □

Example 3.14. *Assume* $\mathcal{A} = \{x, y, z, w\}$ *be a set and* $(\mathcal{L}, \rightarrowtail, 1)$ *be an L-algebra as in Example 3.7. Define* $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$ *by* $\mathfrak{f}(x) = a$, $\mathfrak{f}(y) = c$, $\mathfrak{f}(z) = b$ *and* $\mathfrak{f}(w) = b$. *Then* $\mathcal{A}_a = \{x\}$, $\mathcal{A}_b = \{z, x, y, w\} = \mathcal{A}_1 = \mathcal{A}$, *and* $\mathcal{A}_c = \{y\}$. *Thus,* $b \sim 1$, *but* a *and* c *have no any relation, where* $\sim$ *is an equivalence relation defined in* (2).

4 Block code on L-algebras

As we note, clearly the relation defined in (2), is an equivalence relation on $\mathcal{L}$. Thus it provides the partition of $\mathcal{L}$. For any $x \in \mathcal{L}$, consider $[x]$ as an equivalence class containing x, which means

$$[x] := \{y \mid y \sim x\} = \{y \mid \mathcal{A}_x = \mathcal{A}_y\} = \{y \in \mathcal{L} \mid \mathfrak{f}(\mathcal{A}) \cap \downarrow x = \mathfrak{f}(\mathcal{A}) \cap \downarrow y\}.$$

Now, we define a binary block code of length n from a finite L-algebra, where $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, let $\mathcal{A} = \{a_1, a_2, \cdots, a_n\}$ and $\mathcal{L}$ be a finite L-algebra. Then every $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$ determines a binary block code C of length n as follows:

Let $x \in \mathcal{L}$. Then for $[x]$ the correspondence code-word is

$$c_x : c_1 c_2 \cdots c_n \quad \text{such that } c_i = \hat{\mathfrak{f}}_x(a_i), \quad \text{where } a_i \in \mathcal{A}.$$

We called C an L-code based on $\mathcal{L}$ and denote it by $C_{\mathcal{L}}$.

Example 4.1. (i) *According to Example 3.4, we have*

$$\mathcal{A}_0 = \{0\},\ \mathcal{A}_a = \{0, x\},\ \mathcal{A}_b = \{0, x, y\},\ \text{and } \mathcal{A}_1 = \{0, x, y, z\}.$$

Then

$\hat{\mathfrak{f}}_x$	0	x	y	z
$\hat{\mathfrak{f}}_0$	1	0	0	0
$\hat{\mathfrak{f}}_a$	1	1	0	0
$\hat{\mathfrak{f}}_b$	1	1	1	0
$\hat{\mathfrak{f}}_1$	1	1	1	1

Thus, the total number code-words is 4 *as follows:*

$$c_0 = 1000,\ c_1 = 1100,\ c_2 = 1110,\ c_3 = 1111.$$

(*ii*) *According to Example 3.11, we have*

$$\mathcal{A}_a = \{x\},\ \mathcal{A}_b = \{x, y\},\ \mathcal{A}_c = \{x, z\},\ \mathcal{A}_1 = \mathcal{A}.$$

Then

$\hat{\mathfrak{f}}_x$	x	y	z	w
$\hat{\mathfrak{f}}_a$	1	0	0	0
$\hat{\mathfrak{f}}_b$	1	1	0	0
$\hat{\mathfrak{f}}_c$	1	0	1	0
$\hat{\mathfrak{f}}_1$	1	1	1	1

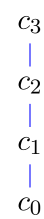
Thus, the total number code-words is 4 *as follows:*

$$c_0 = 1000,\ c_1 = 1100,\ c_2 = 1010,\ c_3 = 1111.$$

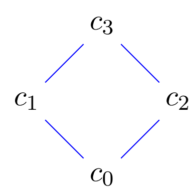
Let $x, y \in \mathcal{L}$ and $C_x = x_1x_2\cdots x_n$ and $C_y = y_1y_2\cdots y_n$ be two code-words belonging to a binary block code C. Define an order relation $\preceq$ on C as follows:

$$C_x \preceq C_y \ \Leftrightarrow\ x_i \leq y_i,\ \forall\ i \in \{1, 2, \cdots, n\}. \tag{3}$$

Example 4.2. (*i*) *According to Example 4.1(i) and* (3)*, clearly we have* $c_0 \preceq c_1 \preceq c_2 \preceq c_3$.

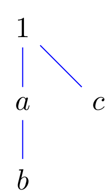


(*ii*) *According to Example 4.1(ii) and* (3)*, clearly we have* $c_0 \preceq c_1, c_2 \preceq c_3$.



So, c_1 *and* c_2 *are incomparable.*

Example 4.3. *Let* $\mathcal{A} = \{x, y, z, w\}$ *be a set and* $(\mathcal{L} = \{a, b, c, 1\}, \leq)$ *be a poset with the following Hasse diagram:*



Then $(\mathcal{L}, \rightarrowtail, 1)$ *is an L-algebra where*

$\rightarrowtail$	a	b	c	1
a	1	a	b	1
b	1	1	a	1
c	a	a	1	1
1	a	b	c	1

Define $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$ *by*

$$\mathfrak{f}(x) = a, \ \mathfrak{f}(y) = \mathfrak{f}(z) = b, \ \mathfrak{f}(w) = 1.$$

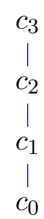
Clearly, $\mathcal{A}_a = \{x, y, z\}$, $\mathcal{A}_b = \{y, z\}$, $\mathcal{A}_c = \emptyset$ *and* $\mathcal{A}_1 = \mathcal{A}$. *Then*

$\hat{\mathfrak{f}}_x$	x	y	z	w
$\hat{\mathfrak{f}}_a$	1	1	1	0
$\hat{\mathfrak{f}}_b$	0	1	1	0
$\hat{\mathfrak{f}}_c$	0	0	0	0
$\hat{\mathfrak{f}}_1$	1	1	1	1

Thus, the total number code-words is 4 *as follows:*

$$c_0 = 0000, \ c_1 = 0110, \ c_2 = 1110, \ c_3 = 1111.$$

By using (3), *we get* $c_0 \preceq c_1 \preceq c_2 \preceq c_3$. *Hence,*



Note. According to Example 4.2, diagrams of code-words are isomorphic with Hasse diagram of L-algebras, but in Example 4.3, diagram of code-words and Hasse diagram of L-algebra are not isomorphic.

Theorem 4.4. *For a finite L-algebra $\mathcal{L}$, there exists a block code C of length n, where $n \in \mathbb{N}$ such that $(\mathcal{L}, \leq)$ is isomorphic $(C, \preceq)$.*

Proof. Since $\mathcal{L}$ is finite, assume $\mathcal{L} = \{l_1, l_2, \cdots, l_n\}$. Let $\mathcal{A} = \mathcal{L}$. Then consider identity map $\mathfrak{f} : \mathcal{A} \to \mathcal{L}$. Clearly, $\mathfrak{f} \in \mathcal{LF}(\mathcal{A})$. Suppose $\dfrac{\mathcal{L}}{\sim} = \{[x] \mid x \in \mathcal{L}\}$. Define $\varphi : \dfrac{\mathcal{L}}{\sim} \to C$ such that $\varphi[x] = c_x$, where $c_x = \hat{\mathfrak{f}}_x(x_1)\hat{\mathfrak{f}}_x(x_2) \cdots \hat{\mathfrak{f}}_x(x_n)$. Suppose $[x] = [y]$. Then

$$\begin{aligned}[x] = [y] \ &\Leftrightarrow \ x \sim y \ \Leftrightarrow \ \mathcal{A}_x = \mathcal{A}_y \ \Leftrightarrow \ \{a \in \mathcal{A} \mid \mathfrak{f}(a) \rightarrowtail x = 1\} = \{b \in \mathcal{A} \mid \mathfrak{f}(b) \rightarrowtail y = 1\} \\ &\Leftrightarrow \ \Big(\hat{\mathfrak{f}}_x(a) = 1 \Leftrightarrow \hat{\mathfrak{f}}_y(b) = 1\Big) \ \Leftrightarrow \ \Big(\hat{\mathfrak{f}}_x(a) = 0 \Leftrightarrow \hat{\mathfrak{f}}_y(b) = 0\Big) \\ &\Leftrightarrow \ c_x = c_y.\end{aligned}$$

So, φ is well-defined, one-to-one and surjective. Now, suppose $x \leq y$. Then by Proposition 3.6, $\mathcal{A}_x \subseteq \mathcal{A}_y$ and so $\hat{\mathfrak{f}}_x(a) \leq \hat{\mathfrak{f}}_y(a)$. Hence, $c_x \preceq c_y$.

If $x \in \mathcal{L}$ and $y \notin \mathcal{A}_x$. Then $\mathfrak{f}(y) \rightarrowtail x \neq 1$, and so $\hat{\mathfrak{f}}_x(y) = 0$. So, $\hat{\mathfrak{f}}_x(y) \leq \hat{\mathfrak{f}}_y(y)$. Hence, in this case $c_x \preceq c_y$. Thus, φ is order preserving. Therefore, $(\mathcal{L}, \leq) \simeq (C, \preceq)$. □

Let $\mathcal{C}_n$ be an arbitrary block-code with n code-words of length n. We consider the matrix $M_{C_n} = (m_{i,j})_{i,j \in \{1,2,\cdots,n\}} \in M_n(\{0,1\})$ with the rows consisting of the code-words of $\mathcal{C}_n$. This matrix is called the matrix associated to the code $\mathcal{C}_n$.

Theorem 4.5. *With the above notations, if the code-word $\underbrace{11\cdots1}_{n-times}$ is in $\mathcal{C}_n$ and the matrix M_C is upper triangular with $m_{ii}=1$, for all $i\in\{1,2,\cdots,n\}$, there are a set $\mathcal{A}$ with n elements, an L-algebra $\mathcal{L}$ and an ℓ-function $\mathfrak{f}:\mathcal{A}\to\mathcal{L}$ such that $\mathfrak{f}$ determines $\mathcal{C}_n$.*

Proof. We consider on $\mathcal{C}_n$ the lexicographic order, denoted by $\leq_{\text{lex}}$. Obviously, $(C,\leq_{\text{lex}})$ is a totally ordered set. Let $C=\{w_1,w_2,\cdots,w_n\}$, with $w_1\leq_{\text{lex}} w_2\leq_{\text{lex}}\cdots\leq_{\text{lex}} w_n$. From here, we obtain that $w_n=\underbrace{11\cdots1}_{n-\text{times}}$ and $w_1=\underbrace{00\cdots0}_{(n-1)-\text{times}}1$. So, w_n is a maximal element and w_1 is a minimal element in $(C,\leq_{\text{lex}})$. We define on $(C,\leq_{\text{lex}})$ a binary operation $\rightarrowtail$ as follows:

$$w_i\rightarrowtail w_j=\begin{cases}1 & w_i\leq_{\text{lex}} w_j\\ w_j & \text{o.w}\end{cases} \tag{4}$$

It results that $\mathcal{L}=(C,\rightarrowtail,w_n)$ becomes an L-algebra and C is isomorphic to C_n as L-algebras. We consider $\mathcal{A}=C$ and the identity map $\mathfrak{f}:\mathcal{A}\to C$, $\mathfrak{f}(w)=w$ as an ℓ-function. The decomposition of $\mathfrak{f}$ provides a family of maps $\Omega=\{\hat{\mathfrak{f}}_x:\mathcal{A}\to\{0,1\}\mid \hat{\mathfrak{f}}_x(a)=1,\ \text{if and only if}\ \mathfrak{f}(a)\rightarrowtail x=1,\forall\ a\in\mathcal{A},\ x\in\mathcal{L}\}$. This family is the binary block-code $\mathcal{C}_n$ relative to the order relation $\leq_{\text{lex}}$. □

Example 4.6. *Let $\mathcal{C}=\{0001,0101,1101,1111\}$ be a binary block codes. Using the lexicographic order, the code $\mathcal{C}$ can be written*

$$c_0=0001,\quad c_1=0101,\quad c_2=1101,\quad c_3=1111.$$

Then by using (4)*, we have*

$\rightarrowtail$	c_0	c_1	c_2	c_3
c_0	c_3	c_3	c_3	c_3
c_1	c_0	c_3	c_3	c_3
c_2	c_0	c_1	c_3	c_3
c_3	c_0	c_1	c_2	c_3

By routine calculation, clearly $(\mathcal{C},\rightarrowtail,c_3)$ is an L-algebra.

We denote by C_n the set of the binary block-codes of the form given in the Theorem 4.5.

Remark 4.7. *Using above technique, we remark that an L-algebra determines a unique binary block-code, but a binary block-code as in Theorem 4.5 can be determined by two or more algebras. If two L-algebras, $\mathcal{L}_1$ and $\mathcal{L}_2$ determine the same binary block-code, we call them code-similar algebras, denoted by $\mathcal{L}_1\bowtie\mathcal{L}_2$.*

Example 4.8. *Let $\mathcal{A}=\{x,y,z,w\}$ be a set. Consider $\mathcal{L}_1$ as an L-algebra as in Example 3.11 and $\mathcal{L}_2$ as an L-algebra as in Example 4.3. Define $\mathfrak{f}_1:\mathcal{A}\to\mathcal{L}_1$ and $\mathfrak{f}_2:\mathcal{A}\to\mathcal{L}_2$ as follows:*

$$\mathfrak{f}_1(x)=a,\ \mathfrak{f}_1(y)=b,\ \textit{and}\ \mathfrak{f}_1(z)=\mathfrak{f}_1(w)=1.$$

and

$$\mathfrak{f}_2(x)=b,\ \mathfrak{f}_2(y)=a,\ \textit{and}\ \mathfrak{f}_2(z)=\mathfrak{f}_2(w)=1.$$

Then

$(\hat{\mathfrak{f}}_1)_x$	x	y	z	w
$(\hat{\mathfrak{f}}_1)_a$	1	0	0	0
$(\hat{\mathfrak{f}}_1)_b$	1	1	0	0
$(\hat{\mathfrak{f}}_1)_c$	0	0	0	0
$(\hat{\mathfrak{f}}_1)_1$	1	1	1	1

$(\hat{\mathfrak{f}}_2)_x$	x	y	z	w
$(\hat{\mathfrak{f}}_2)_a$	1	0	0	0
$(\hat{\mathfrak{f}}_2)_b$	1	1	0	0
$(\hat{\mathfrak{f}}_2)_c$	0	0	0	0
$(\hat{\mathfrak{f}}_2)_1$	1	1	1	1

Since the code-word 0000 *can be ignored, it does not affect the final result, we get $\mathcal{L}_1\bowtie\mathcal{L}_2$.*

Remark 4.9. *If we consider $\mathcal{L}_n$, the set of all finite L-algebras with n elements, then the relation code-similar is an equivalence relation on $\mathcal{L}_n$. It means that for any $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{L}_n$, we have*

$$\mathcal{L}_1 \bowtie \mathcal{L}_2 \Leftrightarrow C_{\mathcal{L}_1} = C_{\mathcal{L}_2},$$

where $C_{\mathcal{L}_1}$ and $C_{\mathcal{L}_2}$ are block-codes corresponding to $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

Let $\mathcal{Q}_n$ be the quotient set. For $C \in C_n$, an equivalent class in $\mathcal{Q}_n$ is

$$C_b = \{B \in \mathcal{L}_n \mid B \text{ determines the binary block-code } C\}.$$

Proposition 4.10. *The quotient set $\mathcal{Q}_n$ has $2^{\frac{(n-1)(n-2)}{2}}$ elements, the same cardinal as the set C_n.*

Proof. We will compute the cardinal of the set C_n. For $C \in C_n$, let M_C be its associated matrix. This matrix is upper triangular with $m_{ii} = 1$, for all $i \in \{1, 2, \cdots, n\}$. We calculate in how many different ways the rows of such a matrix can be written. The second row of the matrix M_C has the form $(0, 1, a_3, \cdots, a_n)$, where $a_3, \cdots, a_n \in \{0, 1\}$. Therefore, the number of different rows of this type is 2^{n-2} and it is equal with the number of functions from a set with $n-2$ elements to the set $\{0, 1\}$. The third row of the matrix M_C has the form $(0, 0, 1, a_4, \cdots, a_n)$, where $a_4, \cdots, a_n \in \{0, 1\}$. In the same way, it results that the number of different rows of this type is 2^{n-3}. Finally, we get that the cardinal of the set C_n is $2^{n-2} \cdot 2^{n-3} \cdots 2 = 2^{\frac{(n-1)(n-2)}{2}}$. □

Remark 4.11. *If $\mathbb{K}_n$ is the number of all finite non-isomorphic L-algebras with n elements, then $\mathbb{K}_n \geq 2^{\frac{(n-1)(n-2)}{2}}$.*

Remark 4.12. *Let $C_1, C_2 \in C_n$ and M_{C_1}, M_{C_2} be the associated matrices. We denote by $r_{j_{C_i}}$ a row in the matrix M_{C_i}, $i \in \{1, 2\}$, $j \in \{1, 2, \cdots, n\}$. On C_n, we define the following totally order relation:*
$C_1 \geq_{lex} C_2$ if there is $i \in \{2, 3, \cdots, n\}$ such that $r_{1_{C_1}} = r_{1_{C_2}}$, $r_{2_{C_1}} = r_{2_{C_2}}$, $\cdots$, $r_{i-1_{C_1}} = r_{i-1_{C_2}}$ and $r_{i_{C_1}} \geq_{lex} r_{i_{C_2}}$ where $\geq_{lex}$ is the lexicographic order.

Proposition 4.13. *Let $X = (a_{i,j})_{i \in \{1,2,\cdots,n\},\ j \in \{1,2,\cdots,m\}} \in M_{n,m}(\{0, 1\})$ be a matrix with rows lexicographic ordered in the descending sense. Starting from this matrix, we can find a matrix $\mathcal{B} = (b_{i,j})_{i,j \in \{1,2,\cdots,q\}} \in M_q(\{0, 1\})$, $q = n + m$, such that $\mathcal{B}$ is an upper triangular matrix, with $b_{ii} = 1$, for all $i \in \{1, 2, \cdots, q\}$ and X becomes a submatrix of the matrix $\mathcal{B}$.*

Proof. We insert in the left side of the matrix X (from the right to the left) the following n new columns of the form $\underbrace{00\cdots1}_{n-\text{times}}, \underbrace{00\cdots10}_{n-\text{times},}, \cdots, \underbrace{10\cdots0}_{n-\text{times},}$. It results a new matrix $\mathcal{D}$ with n rows and q columns.

Now, we insert in the bottom of the matrix $\mathcal{D}$ the following m rows:

$$\underbrace{00\cdots0}_{n-\text{times},}\underbrace{10\cdots0}_{m-\text{times}}, \quad \underbrace{00\cdots0}_{(n+1)-\text{times},}\ \underbrace{01\cdots0}_{(m-1)-\text{times}}\ , \ \cdots, \quad \underbrace{00\cdots0}_{(n+m-1)-\text{times}}\ 1.$$

We obtained the asked matrix $\mathcal{B}$. □

Theorem 4.14. *With the above notations, we consider C be a binary block code with n code-words of length m, $n \neq m$, or a block-code with n code-words of length n such that the code-word $\underbrace{11\cdots1}_{n-times}$ is not in C, or a block-code with n code-words of length n such that the matrix M_C is not upper triangular. There are a natural number $q \geq \max\{m, n\}$, a set $\mathcal{A}$ with m elements and an ℓ-function $\mathfrak{f} : \mathcal{A} \to C_q$ such that the obtained block-code $\mathcal{C}_q$ contains the block-code C as a subset.*

Proof. Let C be a binary block-code, $C = \{w_1, w_2, \cdots, w_n\}$, with code-words of length m. We consider the code-words $w_1, w_2, \cdots, w_n$ lexicographic ordered, $w_1 \geq_{\text{lex}} w_2 \geq_{\text{lex}} \cdots \geq_{\text{lex}} w_n$. Let $M \in M_{n,m}(\{0,1\})$ be the associated matrix with the rows $w_1, \cdots, w_n$ in this order. Using Proposition 4.12, we can extend the matrix M to a square matrix $M' \in M_q(\{0,1\})$, $q = m+n$, such that $M' = (m'_{i,j})_{i,j\in\{1,2,\cdots,q\}}$ is an upper triangular matrix with $m_{ii} = 1$, for all $i \in \{1, 2, \cdots, q\}$. If the first line of the matrix M' is not $\underbrace{11\cdots1}_{q-\text{times}}$ then we insert the row $\underbrace{11\cdots1}_{(q+1)-\text{times}}$ as a first row and the column $1\underbrace{00\cdots0}_{q-\text{times}}$ as a first column . Applying Theorem 4.5, for the matrix M', we obtain an L-algebra $C_q = \{x_1, \cdots, x_q\}$, with $x_1 = 1$ the greatest element of the algebra C_q and a binary block-code C_{C_q}. Assuming that the initial columns of the matrix M have in the new matrix M' positions $i_{j_1}, i_{j_2}, \cdots, i_{j_m} \in \{1, 2, \cdots, q\}$, let $\mathcal{A} = \{x_{j_1}, x_{j_2}, \cdots, x_{j_m}\} \subseteq C_q$. The ℓ-function $\mathfrak{f} : \mathcal{A} \to C_q$, $\mathfrak{f}(x_{j_i}) = x_{j_i}$, $i \in \{1, 2, \cdots, m\}$, determines the binary block-code C_{C_q} such that $C \subseteq C_{C_q}$. □

Example 4.15. *Let $\mathcal{C} = \{00100, 01100, 11100, 11101\}$ be a binary block code. Using the lexicographic order, the code $\mathcal{C}$ can be written $\mathcal{C} = \{11101, 11100, 01100, 00100\} = \{w_1, w_2, w_3, w_4\}$. Let $M_{\mathcal{C}} \in M_{4,5}(\{0,1\})$ be the associated matrix,*

$$M_{\mathcal{C}} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Using Proposition 4.13, we construct an upper triangular matrix, starting from the matrix $M_{\mathcal{C}}$. It results the following matrices:

$$\mathcal{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & \vdots & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & \vdots & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & \vdots & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & \vdots & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \vdots & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since the first row is not $\underbrace{11\cdots1}_{9}$, *using Theorem 4.14, we insert a new row* $\underbrace{11\cdots1}_{10}$ *as a first row and a new column* $\underbrace{10\cdots0}_{10}$ *as a first column. We obtain the following matrix:*

$$\mathcal{B}' = \begin{pmatrix}
1 & \vdots & 1 & 1 & 1 & 1 & \vdots & 1 & 1 & 1 & 1 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
0 & \vdots & 1 & 0 & 0 & 0 & \vdots & 1 & 1 & 1 & 0 & 1 \\
0 & \vdots & 0 & 1 & 0 & 0 & \vdots & 1 & 1 & 1 & 0 & 0 \\
0 & \vdots & 0 & 0 & 1 & 0 & \vdots & 0 & 1 & 1 & 0 & 0 \\
0 & \vdots & 0 & 0 & 0 & 1 & \vdots & 0 & 0 & 1 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
0 & \vdots & 0 & 0 & 0 & 0 & \vdots & 1 & 0 & 0 & 0 & 0 \\
0 & \vdots & 0 & 0 & 0 & 0 & \vdots & 0 & 1 & 0 & 0 & 0 \\
0 & \vdots & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 1 & 0 & 0 \\
0 & \vdots & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 1 & 0 \\
0 & \vdots & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

The binary block-code $\mathcal{W} = \{w_1, \cdots, w_{10}\}$, *whose code-words are the rows of the matrix* $\mathcal{B}'$, *determines an L-algebra* $(\mathcal{W}, \rightarrowtail, w_1)$ *by using* (4). *Let* $\mathcal{A} = \{w_6, w_7, w_8, w_9, w_{10}\}$ *and* $\mathfrak{f} : \mathcal{A} \to \mathcal{W}$, $\mathfrak{f}(w_i) = w_i$, $i \in \{6, 7, 8, 9, 10\}$ *be an* ℓ*-function which determines the binary block-code*

$$\mathcal{U} = \{11111, 11101, 11100, 01100, 00100, 10000, 01000, 00100, 00010, 00001\}.$$

The code $\mathcal{C}$ *is a subset of the code* $\mathcal{U}$.

Conclusion

By using the notion of L-algebras, the notions of block code, x-function and x-subsets on an arbitrary L-algebra are defined. Then some related properties and examples are provided. Also, by using these notions, an equivalence relation on L-algebra and a new order on the generated code based on L-algebras are introduced. Finally, a method is provided which allows us to find an L-algebra starting from a given binary block code. The main motivation of this work is that by using an algebraic structure L-algebras, it is possible to create block codes, and vice versa, by using arbitrary given binary codes and using the lexicographic relation to create an L-algebra, that is, a binary operation of the definition to satisfy the characteristics of L-algebra.

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