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Positive implicative BE-filters of BE-algebras based on Łukasiewicz fuzzy sets

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Abstract

Łukasiewicz fuzzy set is applied to positive implicative filter of BE-algebra. The notion of positive implicative Łukasiewicz fuzzy BE-filters is introduced, and its properties are investigated. The relationship between fuzzy positive implicative BE-filter and positive implicative Łukasiewicz fuzzy BE-filter is discussed, and conditions under which Łukasiewicz fuzzy BE-filter can be positive implicative Łukasiewicz fuzzy BE-filter are explored. Characterizations of positive implicative Łukasiewicz fuzzy BE-filter are provided. Conditions for Łukasiewicz fuzzy set to be positive implicative Łukasiewicz fuzzy BE-filter are considered. Conditions are found where $\in$-set, q-set, and O-set of the Łukasiewicz fuzzy set can be positive implicative BE-filter.

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1 Introduction

BCK-algebra and BCI-algebra are introduced by Y. Imai, K. Iséki and S. Tanaka in 1966 as algebraic structures of universal algebra which describe fragments of propositional calculus related to implications known as BCK and BCI-logic. After that, various generalizations were attempted, and as a result, BCC-algebra, BCH-algebra, BE-algebra, BH-algebra, and d-algebra etc. were appeared. In 2008, S. S. Ahn and K. S. So studied ideal theory in BE-algebras (see [2]), and its fuzzy set theory is studied by Y. B. Jun, K. J. Lee and S. Z. Song (see [6]). M. Sambasiva Rao [10] studied positive implicative BE-filters of BE-algebras. Łukasiewicz logic is the logic of the Łukasiewicz t-norm, and it is a non-classical and many-valued logic. It was originally defined in the early 20th century by Jan Łukasiewicz as a three-valued logic. Using the idea of Łukasiewicz t-norm, Y. B. Jun [4] constructed the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and applied it to BCK-algebras and BCI-algebras. S. Z. Song and Y. B. Jun [11] studied Łukasiewicz fuzzy positive implicative ideals in BCK-algebras. S. S. Ahn et al. [1], and A. Rezaei and A. Borumand Saeid [9] studied fuzzy BE-algebras. G. Dymek and A. Walendziak [3] developed the theory of fuzzy filters in BE-algebras. Y. B. Jun and S. S. Ahn applied the Łukasiewicz fuzzy set to BE-filters and subalgebras (see [5]).

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In this paper, we apply the concept of Łukasiewicz fuzzy sets to positive implicative BE-filters of BE-algebras. We introduce the notion of positive implicative Łukasiewicz fuzzy BE-filters, and investigate several properties. We discuss the relationship between fuzzy positive implicative BE-filter and positive implicative Łukasiewicz fuzzy BE-filter. We derive the conditions under which Łukasiewicz fuzzy BE-filter can be positive implicative Łukasiewicz fuzzy BE-filter. We provide characterizations of positive implicative Łukasiewicz fuzzy BE-filter. We explore conditions for Łukasiewicz fuzzy set to be positive implicative Łukasiewicz fuzzy BE-filter. We find the conditions under which $\in$-set, q-set, and O-set of the Łukasiewicz fuzzy set can be positive implicative BE-filter. For the various terms that appear in this paper, we use abbreviated expressions. The list of acronyms is given in Table 1.

Table 1: List of acronyms

Acronyms	Representation
fBE-algebra	fuzzy BE-algebra
fBE-filter	fuzzy BE-filter
piBE-filter	positive implicative BE-filter
fpiBE-filter	fuzzy positive implicative BE-filter
Łf-set	Łukasiewicz fuzzy set
ŁfBE-algebra	Łukasiewicz fuzzy BE-algebra
ŁfBE-filter	Łukasiewicz fuzzy BE-filter
piŁfBE-filter	positive implicative Łukasiewicz fuzzy BE-filter

2 Preliminaries

A *BE-algebra* (see [7]) is a structure $(X;*,1)$ where X is a set together with a binary operation "$*$" and a special element "1" satisfying the conditions:

(BE1) $(\forall a \in X)\ (a * a = 1)$,
(BE2) $(\forall a \in X)\ (a * 1 = 1)$,
(BE3) $(\forall a \in X)\ (1 * a = a)$,
(BE4) $(\forall a, b, c \in X)\ (a * (b * c) = b * (a * c))$.

The order relation "$\leq$" in a BE-algebra $(X;*,1)$ is defined as follows:

$$(\forall a, b \in X)(a \leq b \ \Leftrightarrow \ a * b = 1). \tag{1}$$

Every BE-algebra $(X;*,1)$ satisfies the following conditions (see [7]):

$$(\forall a, b \in X)\,(a * (b * a) = 1)\,. \tag{2}$$
$$(\forall a, b \in X)\,(a * ((a * b) * b) = 1)\,. \tag{3}$$

Let $(X;*,1)$ be a BE-algebra. A subset A of X is called

- a *BE-subalgebra* of $(X;*,1)$ if it satisfies:

$$(\forall a, b \in A)(a * b \in A), \tag{4}$$

- a *BE-filter* of $(X;*,1)$ (see [7]) if it satisfies:

$$1 \in A, \tag{5}$$
$$(\forall a, b \in X)(a * b \in A,\ a \in A \ \Rightarrow \ b \in A). \tag{6}$$

- a *piBE-filter* of $(X;*,1)$ (see [10]) if it satisfies (5) and

$$(\forall a, b, c \in X)(a * ((b * c) * b) \in A,\ a \in A \ \Rightarrow \ b \in A). \tag{7}$$

Let $(X;*,1)$ be a BE-algebra. A fuzzy set ξ in X is called

- a *fBE-algebra* of $(X;*,1)$ (see [1]) if it satisfies:

$$(\forall a,b\in X)(\xi(a*b)\geq \min\{\xi(a),\xi(b)\}). \tag{8}$$

- a *fBE-filter* of $(X;*,1)$ (see [3]) if it satisfies:

$$(\forall a\in X)(\xi(1)\geq \xi(a)), \tag{9}$$
$$(\forall a,b\in X)(\xi(b)\geq \min\{\xi(a*b),\xi(a)\}). \tag{10}$$

- a *fpiBE-filter* of $(X;*,1)$ (see [10]) if it satisfies (9) and

$$(\forall a,b,c\in X)(\xi(b)\geq \min\{\xi(a*((b*c)*b)),\xi(a)\}). \tag{11}$$

A fuzzy set ξ in a set X of the form

$$\xi(b):=\begin{cases} t\in(0,1] & \text{if } b=a,\\ 0 & \text{if } b\neq a,\end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $\langle a/t\rangle$.

For a fuzzy set ξ in a set X, we say that a fuzzy point $\langle a/t\rangle$ is

(i) *contained* in ξ, denoted by $\langle a/t\rangle\in\xi$, (see [8]) if $\xi(a)\geq t$.

(ii) *quasi-coincident* with ξ, denoted by $\langle a/t\rangle\, q\,\xi$, (see [8]) if $\xi(a)+t>1$.

If $\langle a/t\rangle\,\alpha\,\xi$ is not established for $\alpha\in\{\in,q\}$, it is denoted by $\langle a/t\rangle\,\overline{\alpha}\,\xi$.

Let ξ be a fuzzy set in a set X and let $\varepsilon\in(0,1)$. A function

$$\overset{\varepsilon}{\xi}: X\to[0,1],\ x\mapsto \max\{0,\xi(x)+\varepsilon-1\}$$

is called the *Łf-set* of ξ in X.

For the Łf-set $\overset{\varepsilon}{\xi}$ of ξ in X and $t\in(0,1]$, consider the sets

$$(\overset{\varepsilon}{\xi},t)_{\in}:=\{x\in X\mid \langle x/t\rangle\in\overset{\varepsilon}{\xi}\},$$
$$(\overset{\varepsilon}{\xi},t)_{q}:=\{x\in X\mid \langle x/t\rangle\, q\,\overset{\varepsilon}{\xi}\},$$

which are called the $\in$*-set* and q*-set*, respectively, of $\overset{\varepsilon}{\xi}$ (with value t). Also, consider a set:

$$O(\overset{\varepsilon}{\xi}):=\{x\in X\mid \overset{\varepsilon}{\xi}(x)>0\} \tag{12}$$

which is called an O*-set* of $\overset{\varepsilon}{\xi}$. It is observed that

$$O(\overset{\varepsilon}{\xi})=\{x\in X\mid \xi(x)+\varepsilon-1>0\}.$$

In what follows, let $(X;*,1)$ and ξ be a BE-algebra and a fuzzy set in X respectively, and ε is an element of $(0,1)$ unless otherwise specified.

Definition 2.1. [5] *A Łf-set $\overset{\varepsilon}{\xi}$ of ξ in X is called a ŁfBE-algebra of $(X;*,1)$ if it satisfies:*

$$\langle x/t_a\rangle\in\overset{\varepsilon}{\xi},\ \langle y/t_b\rangle\in\overset{\varepsilon}{\xi}\ \Rightarrow\ \langle (x*y)/\min\{t_a,t_b\}\rangle\in\overset{\varepsilon}{\xi} \tag{13}$$

for all $x,y\in X$ and $t_a,t_b\in(0,1]$.

Definition 2.2. [5] *A Łf-set $\overset{\varepsilon}{\xi}$ of ξ in X is called a ŁfBE-filter of $(X;*,1)$ if it satisfies:*

$$(\forall x\in X)(\forall t_a\in(0,1])\left(x\in(\overset{\varepsilon}{\xi},t_a)_{\in}\ \Rightarrow\ 1\in(\overset{\varepsilon}{\xi},t_a)_{\in}\right), \tag{14}$$

$$(\forall x,y\in X)(\forall t_a,t_b\in(0,1])\begin{pmatrix} x*y\in(\overset{\varepsilon}{\xi},t_b)_{\in},\ x\in(\overset{\varepsilon}{\xi},t_a)_{\in}\\ \Rightarrow\ y\in(\overset{\varepsilon}{\xi},\min\{t_a,t_b\})_{\in}\end{pmatrix}. \tag{15}$$

3 Positive implicative Łukasiewicz fuzzy BE-filters

Definition 3.1. *A Łf-set* $\overset{\varepsilon}{\xi}$ *of* ξ *in* X *is called a piŁfBE-filter of* $(X;*,1)$ *if it satisfies* (14) *and*

$$\left(\begin{array}{l} \forall x,y,z \in X, \\ \forall t_a,t_b \in (0,1] \end{array}\right)\left(\begin{array}{l} x \in (\overset{\varepsilon}{\xi},t_a)_{\in},\ x*((y*z)*y) \in (\overset{\varepsilon}{\xi},t_b)_{\in} \\ \quad \Rightarrow\ y \in (\overset{\varepsilon}{\xi},\min\{t_a,t_b\})_{\in} \end{array}\right). \tag{16}$$

Example 3.2. *Consider a set* $X=\{1,a,b,c,d\}$*, and define a binary operation "$*$" by Table 2.*

Table 2: Cayley table for the binary operation "$*$"

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

Then $(X;*,1)$ *is a BE-algebra (see [10]). Define a fuzzy set* ξ *in* X *as follows:*

$$\xi : X \to [0,1],\ x \mapsto \begin{cases} 0.77 & \text{if } x=1, \\ 0.38 & \text{if } x=a, \\ 0.77 & \text{if } x=b, \\ 0.77 & \text{if } x=c, \\ 0.52 & \text{if } x=d. \end{cases}$$

Given $\varepsilon := 0.46$*, the Łf-set* $\overset{\varepsilon}{\xi}$ *of* ξ *in* X *is given as follows:*

$$\overset{\varepsilon}{\xi} : X \to [0,1],\ x \mapsto \begin{cases} 0.23 & \text{if } x \in \{1,b,c\}, \\ 0.00 & \text{otherwise.} \end{cases}$$

It is routine to verify that $\overset{\varepsilon}{\xi}$ *is a piŁfBE-filter of* $(X;*,1)$.

We discuss the relationship between a fpiBE-filter and a piŁfBE-filter.

Theorem 3.3. *If* ξ *is a fpiBE-filter of* $(X;*,1)$*, then its Łf-set* $\overset{\varepsilon}{\xi}$ *is a piŁfBE-filter of* $(X;*,1)$.

Proof. Assume that ξ is a fpiBE-filter of $(X;*,1)$ and let $\overset{\varepsilon}{\xi}$ be its Łf-set in X. Let $x \in X$ and $t \in (0,1]$ be such that $x \in (\overset{\varepsilon}{\xi},t)_{\in}$. Then

$$\overset{\varepsilon}{\xi}(1) = \max\{0, \xi(1)+\varepsilon-1\} \geq \max\{0,\xi(x)+\varepsilon-1\} = \overset{\varepsilon}{\xi}(x) \geq t,$$

and so $1 \in (\overset{\varepsilon}{\xi},t)_{\in}$. Let $x,y,z \in X$ and $t_a,t_b \in (0,1]$ be such that

$$x*((y*z)*y) \in (\overset{\varepsilon}{\xi},t_b)_{\in} \text{ and } x \in (\overset{\varepsilon}{\xi},t_a)_{\in}.$$

Then $\overset{\varepsilon}{\xi}(x*((y*z)*y)) \geq t_b$ and $\overset{\varepsilon}{\xi}(x) \geq t_a$, which imply that

$$\begin{aligned} \overset{\varepsilon}{\xi}(y) &= \max\{0,\xi(y)+\varepsilon-1\} \\ &\geq \max\{0,\min\{\xi(x*((y*z)*y)),\xi(x)\}+\varepsilon-1\} \\ &= \max\{0,\min\{\xi(x*((y*z)*y))+\varepsilon-1,\xi(x)+\varepsilon-1\}\} \\ &= \min\{\max\{0,\xi(x*((y*z)*y))+\varepsilon-1\},\max\{0,\xi(x)+\varepsilon-1\}\} \\ &= \min\{\overset{\varepsilon}{\xi}(x*((y*z)*y)),\overset{\varepsilon}{\xi}(x)\} \geq \min\{t_a,t_b\}. \end{aligned}$$

Hence $\langle y/\min\{t_a,t_b\}\rangle \in \overset{\varepsilon}{\xi}$, that is, $y \in (\overset{\varepsilon}{\xi},\min\{t_a,t_b\})_{\in}$. Therefore $\overset{\varepsilon}{\xi}$ is a piŁfBE-filter of $(X;*,1)$. □

In Example 3.2, the fuzzy set ξ is not a fpiBE-filter of $(X;*,1)$ since

$$\xi(a)=0.38 \ngeq 0.52=\min\{\xi(d),\xi(d*((a*b)*a))\}.$$

This shows that the converse of Theorem 3.3 is generally not true, that is, there exists a fuzzy set ξ in X so that its Łf-set ${}_{\xi}^{\varepsilon}$ is a piŁfBE-filter of $(X;*,1)$, but ξ is not a fpiBE-filter of $(X;*,1)$.

If we use the special element "1" instead of "z" in (16) and use (BE1) and (BE2), then we can induce (15). So we know that every piŁfBE-filter is a ŁfBE-filter. But the converse is generally not true as seen in the following example.

Example 3.4. *Consider the BE-algebra $(X;*,1)$ in Example 3.2, and let ξ be a fuzzy set in X given as follows:*

$$\xi : X \to [0,1],\ x \mapsto \begin{cases} 0.97 & \text{if } x=1,\\ 0.79 & \text{if } x=a,\\ 0.45 & \text{if } x=b,\\ 0.56 & \text{if } x=c,\\ 0.38 & \text{if } x=d.\end{cases}$$

If we give $\varepsilon := 0.43$, then the Łf-set ${}_{\xi}^{\varepsilon}$ of ξ in X is calculated as follows:

$${}_{\xi}^{\varepsilon} : X \to [0,1],\ x \mapsto \begin{cases} 0.40 & \text{if } x=1,\\ 0.22 & \text{if } x=a,\\ 0.00 & \text{otherwise.}\end{cases}$$

It is routine to verify that ${}_{\xi}^{\varepsilon}$ is a ŁfBE-filter of $(X;*,1)$. But it is not a piŁfBE-filter of $(X;*,1)$ since $a \in ({}_{\xi}^{\varepsilon}, 0.19)_{\in}$ and $a*((b*c)*b) \in ({}_{\xi}^{\varepsilon}, 0.37)_{\in}$, but $b \notin ({}_{\xi}^{\varepsilon}, \min\{0.19, 0.37\})_{\in}$.

We derive the conditions under which a ŁfBE-filter can be a piŁfBE-filter.

Theorem 3.5. *Let ${}_{\xi}^{\varepsilon}$ be a ŁfBE-filter of $(X;*,1)$. Then the following assertions are equivalent.*

(i) *${}_{\xi}^{\varepsilon}$ is a piŁfBE-filter of $(X;*,1)$.*

(ii) *${}_{\xi}^{\varepsilon}$ satisfies:*

$$(\forall x,y \in X)(\forall t \in (0,1])\left((x*y)*x \in ({}_{\xi}^{\varepsilon}, t)_{\in} \ \Rightarrow\ x \in ({}_{\xi}^{\varepsilon}, t)_{\in}\right). \tag{17}$$

(iii) *${}_{\xi}^{\varepsilon}$ satisfies:*

$$\begin{array}{l}(\forall x,y,z \in X)(\forall t_a,t_b \in (0,1])\\ \left(\begin{array}{l} z \in ({}_{\xi}^{\varepsilon}, t_b)_{\in},\ (x*y)*(z*x) \in ({}_{\xi}^{\varepsilon}, t_a)_{\in}\\ \quad \Rightarrow\ x \in ({}_{\xi}^{\varepsilon}, \min\{t_a,t_b\})_{\in}\end{array}\right).\end{array} \tag{18}$$

Proof. (i) ⇒ (ii). Assume that ${}_{\xi}^{\varepsilon}$ is a piŁfBE-filter of $(X;*,1)$. Let $x,y \in X$ and $t \in (0,1]$ be such that $(x*y)*x \in ({}_{\xi}^{\varepsilon}, t)_{\in}$. Then

$$1*((x*y)*x)=(x*y)*x \in ({}_{\xi}^{\varepsilon}, t)_{\in}$$

by (BE3). Since $1 \in ({}_{\xi}^{\varepsilon}, {}_{\xi}^{\varepsilon}(1))_{\in}$, it follows from (14) and (16) that

$$x \in ({}_{\xi}^{\varepsilon}, \min\{t, {}_{\xi}^{\varepsilon}(1)\})_{\in} = ({}_{\xi}^{\varepsilon}, t)_{\in}.$$

(ii) ⇒ (iii). Suppose that ${}_{\xi}^{\varepsilon}$ satisfies (17) and let $x,y,b \in X$ and $t_a,t_b \in (0,1]$ be such that $b \in ({}_{\xi}^{\varepsilon}, t_b)_{\in}$ and $(x*y)*(b*x) \in ({}_{\xi}^{\varepsilon}, t_a)_{\in}$. Then

$$b*((x*y)*x)=(x*y)*(b*x) \in ({}_{\xi}^{\varepsilon}, t_a)_{\in}$$

by (BE4), which implies from (15) that $(x*y)*x \in ({}_{\xi}^{\varepsilon}, \min\{t_a,t_b\})_{\in}$. Hence $x \in ({}_{\xi}^{\varepsilon}, \min\{t_a,t_b\})_{\in}$ by (17).

(iii) ⇒ (i). Suppose that ${}_{\xi}^{\varepsilon}$ satisfies (18) and let $x,y,z \in X$ and $t_a,t_b \in (0,1]$ be such that $x \in ({}_{\xi}^{\varepsilon}, t_a)_{\in}$ and $x*((y*z)*y) \in ({}_{\xi}^{\varepsilon}, t_b)_{\in}$. Then

$$(y*z)*(x*y)=x*((y*z)*y) \in ({}_{\xi}^{\varepsilon}, t_b)_{\in}$$

by (BE4). Using (18) leads to $y \in ({}_{\xi}^{\varepsilon}, \min\{t_a,t_b\})_{\in}$ Therefore ${}_{\xi}^{\varepsilon}$ is a piŁfBE-filter of $(X;*,1)$. □

Theorem 3.6. *A Łf-set ${}_{\xi}^{\varepsilon}$ in X is a piŁfBE-filter of $(X;*,1)$ if and only if it satisfies:*

$$(\forall x \in X)\left({}_{\xi}^{\varepsilon}(1) \geq {}_{\xi}^{\varepsilon}(x)\right), \tag{19}$$

$$(\forall x, y, z \in X)\left({}_{\xi}^{\varepsilon}(y) \geq \min\{{}_{\xi}^{\varepsilon}(x), {}_{\xi}^{\varepsilon}(x * ((y * z) * y))\}\right). \tag{20}$$

Proof. Assume that ${}_{\xi}^{\varepsilon}$ is a piŁfBE-filter of $(X;*,1)$. Since $x \in ({}_{\xi}^{\varepsilon}, {}_{\xi}^{\varepsilon}(x))_{\in}$ for all $x \in X$, it follows from (14) that $1 \in ({}_{\xi}^{\varepsilon}, {}_{\xi}^{\varepsilon}(x))_{\in}$. Hence ${}_{\xi}^{\varepsilon}(1) \geq {}_{\xi}^{\varepsilon}(x)$ for all $x \in X$. Since $x * ((y * z) * y) \in ({}_{\xi}^{\varepsilon}, {}_{\xi}^{\varepsilon}(x * ((y * z) * y)))_{\in}$ and $x \in ({}_{\xi}^{\varepsilon}, {}_{\xi}^{\varepsilon}(x))_{\in}$ for all $x, y, z \in X$, we have

$$y \in ({}_{\xi}^{\varepsilon}, \min\{{}_{\xi}^{\varepsilon}(x * ((y * z) * y)), {}_{\xi}^{\varepsilon}(x)\})_{\in}$$

by (16). Hence ${}_{\xi}^{\varepsilon}(y) \geq \min\{{}_{\xi}^{\varepsilon}(x * ((y * z) * y)), {}_{\xi}^{\varepsilon}(x)\}$ for all $x, y, z \in X$.

Conversely, suppose that ${}_{\xi}^{\varepsilon}$ satisfies (19) and (20). Let $x \in X$ and $t \in (0,1]$ be such that $x \in ({}_{\xi}^{\varepsilon}, t)_{\in}$. Then ${}_{\xi}^{\varepsilon}(1) \geq {}_{\xi}^{\varepsilon}(x) \geq t$ by (19), and so $1 \in ({}_{\xi}^{\varepsilon}, t)_{\in}$. Assume that $x * ((y * z) * y) \in ({}_{\xi}^{\varepsilon}, t_b)_{\in}$ and $x \in ({}_{\xi}^{\varepsilon}, t_a)_{\in}$ for all $x, y, z \in X$ and $t_a, t_b \in (0,1]$. Then ${}_{\xi}^{\varepsilon}(x * ((y * z) * y)) \geq t_b$ and ${}_{\xi}^{\varepsilon}(x) \geq t_a$ It follows from (20) that

$${}_{\xi}^{\varepsilon}(y) \geq \min\{{}_{\xi}^{\varepsilon}(x * ((y * z) * y)), {}_{\xi}^{\varepsilon}(x)\} \geq \min\{t_a, t_b\},$$

i.e., $\langle y/\min\{t_a, t_b\}\rangle \in {}_{\xi}^{\varepsilon}$. Hence $y \in ({}_{\xi}^{\varepsilon}, \min\{t_a, t_b\})_{\in}$. Therefore ${}_{\xi}^{\varepsilon}$ is a piŁfBE-filter of $(X;*,1)$. □

Theorem 3.7. *Let ξ be a fuzzy set in X that satisfies:*

$$(\forall x, y, z \in X)(\xi(x) \geq \min\{\xi(z), \xi((x * y) * (z * x))\}). \tag{21}$$

If a Łf-set ${}_{\xi}^{\varepsilon}$ of ξ in X satisfies (14)*, then it is a piŁfBE-filter of* $(X;*,1)$.

Proof. Let ξ be a fuzzy set in X satisfying the condition (21), and assume that ${}_{\xi}^{\varepsilon}$ satisfies (14). Let $x, y, z \in X$ and $t_a, t_b \in (0,1]$ be such that

$$x * ((y * z) * y) \in ({}_{\xi}^{\varepsilon}, t_b)_{\in} \text{ and } x \in ({}_{\xi}^{\varepsilon}, t_a)_{\in}.$$

Then ${}_{\xi}^{\varepsilon}(x) = \max\{0, \xi(x) + \varepsilon - 1\} \geq t_a$ and

$${}_{\xi}^{\varepsilon}(x * ((y * z) * y)) = \max\{0, \xi(x * ((y * z) * y)) + \varepsilon - 1\} \geq t_b.$$

Since $t_a > 0$ and $t_b > 0$, we have $\xi(x) + \varepsilon - 1 \geq t_a$ and

$$\xi((y * z) * (x * y)) + \varepsilon - 1 = \xi(x * ((y * z) * y)) + \varepsilon - 1 \geq t_b.$$

It follows from (BE4) and (21) that

$$\begin{aligned} \xi(y) + \varepsilon - 1 &\geq \min\{\xi(x),\, \xi((y * z) * (x * y))\} + \varepsilon - 1 \\ &= \min\{\xi(x),\, \xi(x * ((y * z) * y))\} + \varepsilon - 1 \\ &= \min\{\xi(x) + \varepsilon - 1,\, \xi(x * ((y * z) * y)) + \varepsilon - 1\} \\ &\geq \min\{t_a,\, t_b\} > 0. \end{aligned}$$

Hence

$${}_{\xi}^{\varepsilon}(y) = \max\{0, \xi(y) + \varepsilon - 1\} = \xi(y) + \varepsilon - 1 \geq \min\{t_a,\, t_b\},$$

and so $y \in ({}_{\xi}^{\varepsilon}, \min\{t_a,\, t_b\})_{\in}$. Therefored ${}_{\xi}^{\varepsilon}$ is a piŁfBE-filter of $(X;*,1)$. □

Corollary 3.8. *If ξ is a fBE-filter of $(X;*,1)$ that satisfies* (21)*, then its Łf-set ${}_{\xi}^{\varepsilon}$ is a piŁfBE-filter of $(X;*,1)$.*

Lemma 3.9. [5] *If ξ is a fBE-filter of $(X;*,1)$, then its Łf-set ${}_{\xi}^{\varepsilon}$ is a piŁfBE-filter of $(X;*,1)$.*

Theorem 3.10. *If ξ is a fBE-filter of $(X;*,1)$ that satisfies:*

$$(\forall x, y \in X)(\xi(x) \geq \xi((x * y) * x)\}), \tag{22}$$

then its Łf-set ${}_{\xi}^{\varepsilon}$ is a piŁfBE-filter of $(X;*,1)$.

Proof. Let ξ be a fBE-filter of $(X;*,1)$ satisfying the condition (22). Then ${}^{\varepsilon}_{\xi}$ satisfies (14) by Lemma 3.9. Using (BE4), (10) and (22) leads to

$$\xi(x) \geq \xi((x*y)*x) \geq \min\{\xi(z), \xi(z*((x*y)*x))\} = \min\{\xi(z), \xi((x*y)*(z*x))\}$$

for all $x, y, z \in X$. Therefore ${}^{\varepsilon}_{\xi}$ is a piŁfBE-filter of $(X;*,1)$ by Theorem 3.7. $\square$

We find the conditions under which $\in$-set, q-set, and O-set of the Łf-set can be piBE-filter.

Theorem 3.11. *Let ${}^{\varepsilon}_{\xi}$ be a Łf-set in X. Then the $\in$-set $({}^{\varepsilon}_{\xi}, t)_{\in}$ of ${}^{\varepsilon}_{\xi}$ is a piBE-filter of $(X;*,1)$ for all $t \in (0.5, 1]$ if and only if ${}^{\varepsilon}_{\xi}$ satisfies:*

$$(\forall x \in X)\left({}^{\varepsilon}_{\xi}(x) \leq \max\{{}^{\varepsilon}_{\xi}(1), 0.5\}\right), \tag{23}$$

$$(\forall x, y, z \in X)\left(\min\{{}^{\varepsilon}_{\xi}(x), {}^{\varepsilon}_{\xi}(x*((y*z)*y))\} \leq \max\{{}^{\varepsilon}_{\xi}(y), 0.5\}\right). \tag{24}$$

Proof. Assume that $({}^{\varepsilon}_{\xi}, t)_{\in}$ is a piBE-filter of $(X;*,1)$ for $t \in (0.5, 1]$. If there exist $a \in X$ such that ${}^{\varepsilon}_{\xi}(a) > \max\{{}^{\varepsilon}_{\xi}(1), 0.5\}$, then ${}^{\varepsilon}_{\xi}(a) \in (0.5, 1]$ and ${}^{\varepsilon}_{\xi}(1) < {}^{\varepsilon}_{\xi}(a)$. Since $a \in ({}^{\varepsilon}_{\xi}, {}^{\varepsilon}_{\xi}(a))_{\in}$, we have $1 \in ({}^{\varepsilon}_{\xi}, {}^{\varepsilon}_{\xi}(a))_{\in}$ by (14) and thus ${}^{\varepsilon}_{\xi}(1) \geq {}^{\varepsilon}_{\xi}(a)$. This is a contradiction, and so ${}^{\varepsilon}_{\xi}(x) \leq \max\{{}^{\varepsilon}_{\xi}(1), 0.5\}$ for all $x \in X$. If the condition (24) is not valid, then there exist $a, b, c \in X$ such that

$$\min\{{}^{\varepsilon}_{\xi}(a), {}^{\varepsilon}_{\xi}(a*((b*c)*b))\} > \max\{{}^{\varepsilon}_{\xi}(b), 0.5\}.$$

If we take $t := \min\{{}^{\varepsilon}_{\xi}(a), {}^{\varepsilon}_{\xi}(a*((b*c)*b))\}$, then $t \in (0.5, 1]$, $\langle a/t\rangle \in {}^{\varepsilon}_{\xi}$ and $\langle (a*((b*c)*b))/t\rangle \in {}^{\varepsilon}_{\xi}$, but $\langle b/t\rangle \,\overline{\in}\, {}^{\varepsilon}_{\xi}$, that is, $a \in ({}^{\varepsilon}_{\xi}, t)_{\in}$ and $a*((b*c)*b) \in ({}^{\varepsilon}_{\xi}, t)_{\in}$, but $b \notin ({}^{\varepsilon}_{\xi}, t)_{\in}$. This is a contradiction, and thus (24) is valid.

Conversely, suppose that ${}^{\varepsilon}_{\xi}$ satisfies (23) and (24), and let $t \in (0.5, 1]$. For every $x \in ({}^{\varepsilon}_{\xi}, t)_{\in}$, we have $t \leq {}^{\varepsilon}_{\xi}(x) \leq \max\{{}^{\varepsilon}_{\xi}(1), 0.5\}$ by (23). Hence ${}^{\varepsilon}_{\xi}(1) \geq t$, and so $1 \in ({}^{\varepsilon}_{\xi}, t)_{\in}$. Let $x, y, z \in X$ and $t \in (0.5, 1]$ be such that $x \in ({}^{\varepsilon}_{\xi}, t)_{\in}$ and $x*((y*z)*y) \in ({}^{\varepsilon}_{\xi}, t)_{\in}$. Then ${}^{\varepsilon}_{\xi}(x) \geq t$ and ${}^{\varepsilon}_{\xi}(x*((y*z)*y)) \geq t$, which imply from (24) that

$$0.5 < t \leq \min\{{}^{\varepsilon}_{\xi}(x), {}^{\varepsilon}_{\xi}(x*((y*z)*y))\} \leq \max\{{}^{\varepsilon}_{\xi}(y), 0.5\}.$$

Hence $\langle y/t\rangle \in {}^{\varepsilon}_{\xi}$, that is, $y \in ({}^{\varepsilon}_{\xi}, t)_{\in}$. Therefore $({}^{\varepsilon}_{\xi}, t)_{\in}$ is a piBE-filter of $(X;*,1)$ for $t \in (0.5, 1]$. $\square$

Theorem 3.12. *If a Łf-set ${}^{\varepsilon}_{\xi}$ in X satisfies:*

$$(\forall x \in X)(\forall t \in (0.5, 1])\left(\langle x/t\rangle \, q \, {}^{\varepsilon}_{\xi} \;\Rightarrow\; \langle 1/t\rangle \in {}^{\varepsilon}_{\xi}\right), \tag{25}$$

$$\left(\begin{array}{c} \forall x, y, z \in X, \\ \forall t_a, t_b \in (0.5, 1] \end{array}\right)\left(\begin{array}{c} \langle (x*((y*z)*y))/t_a\rangle \, q \, {}^{\varepsilon}_{\xi},\ \langle x/t_b\rangle \, q \, {}^{\varepsilon}_{\xi} \\ \Rightarrow\ \langle y/\max\{t_a, t_b\}\rangle \in {}^{\varepsilon}_{\xi} \end{array}\right), \tag{26}$$

then the non-empty $\in$-set $({}^{\varepsilon}_{\xi}, \max\{t_a, t_b\})_{\in}$ of ${}^{\varepsilon}_{\xi}$ is a piBE-filter of $(X;*,1)$ for all $t_a, t_b \in (0.5, 1]$.

Proof. Assume that ${}^{\varepsilon}_{\xi}$ satisfies (25) and (26). If the $\in$-set $({}^{\varepsilon}_{\xi}, \max\{t_a, t_b\})_{\in}$ of ${}^{\varepsilon}_{\xi}$ is non-empty for all $t_a, t_b \in (0.5, 1]$, then there exists $x \in ({}^{\varepsilon}_{\xi}, \max\{t_a, t_b\})_{\in}$, and so ${}^{\varepsilon}_{\xi}(x) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$, i.e., $\langle x/\max\{t_a, t_b\}\rangle \, q \, {}^{\varepsilon}_{\xi}$. Hence $\langle 1/\max\{t_a, t_b\}\rangle \in {}^{\varepsilon}_{\xi}$ by (25), and thus $1 \in ({}^{\varepsilon}_{\xi}, \max\{t_a, t_b\})_{\in}$. Let $x, y, z \in X$ be such that $x*((y*z)*y) \in ({}^{\varepsilon}_{\xi}, \max\{t_a, t_b\})_{\in}$ and $x \in ({}^{\varepsilon}_{\xi}, \max\{t_a, t_b\})_{\in}$. Then

$${}^{\varepsilon}_{\xi}(x*((y*z)*y)) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$$

and ${}^{\varepsilon}_{\xi}(x) \geq \max\{t_a, t_b\} > 1 - \max\{t_a, t_b\}$, that is,

$$\langle (x*((y*z)*y))/\max\{t_a, t_b\}\rangle \, q \, {}^{\varepsilon}_{\xi} \text{ and } \langle x/\max\{t_a, t_b\}\rangle \, q \, {}^{\varepsilon}_{\xi}.$$

It follows from (26) that $\langle y/\max\{t_a, t_b\}\rangle \in {}^{\varepsilon}_{\xi}$. Hence $y \in ({}^{\varepsilon}_{\xi}, \max\{t_a, t_b\})_{\in}$, and therefore $({}^{\varepsilon}_{\xi}, \max\{t_a, t_b\})_{\in}$ is a piBE-filter of $(X;*,1)$ for all $t_a, t_b \in (0.5, 1]$. $\square$

Theorem 3.13. *If a Łf-set* $\overset{\varepsilon}{\xi}$ *in* X *satisfies* (25) *and*

$$\left(\begin{array}{l}\forall x, y, z \in X, \\ \forall t_a, t_b \in (0.5, 1]\end{array}\right)\left(\begin{array}{l}\langle (x * ((y * z) * y))/t_a\rangle \, q \, \overset{\varepsilon}{\xi}, \ \langle x/t_b\rangle \, q \, \overset{\varepsilon}{\xi} \\ \quad \Rightarrow \ \langle y/\min\{t_a, t_b\}\rangle \in \overset{\varepsilon}{\xi}\end{array}\right), \tag{27}$$

then the non-empty $\in$*-set* $(\overset{\varepsilon}{\xi}, \min\{t_a, t_b\})_{\in}$ *of* $\overset{\varepsilon}{\xi}$ *is a piBE-filter of* $(X; *, 1)$ *for all* $t_a, t_b \in (0.5, 1]$.

Proof. This can be verified through the same process as the proof of Theorem 3.12. □

Theorem 3.14. *If* $\overset{\varepsilon}{\xi}$ *is a piŁfBE-filter of* $(X; *, 1)$*, then its q-set* $(\overset{\varepsilon}{\xi}, t)_q$ *is a piBE-filter of* $(X; *, 1)$ *for all* $t \in (0, 1]$.

Proof. Let $\overset{\varepsilon}{\xi}$ be a piŁfBE-filter of $(X; *, 1)$ and let $t \in (0, 1]$. If $1 \notin (\overset{\varepsilon}{\xi}, t)_q$, then $\langle 1/t\rangle \, \overline{q} \, \overset{\varepsilon}{\xi}$, i.e., $\overset{\varepsilon}{\xi}(1) + t \leq 1$. Since $\langle x/\overset{\varepsilon}{\xi}(x)\rangle \in \overset{\varepsilon}{\xi}$ for all $x \in X$, we get $\langle 1/\overset{\varepsilon}{\xi}(x)\rangle \in \overset{\varepsilon}{\xi}$ for all $x \in X$ by (14). Hence $\overset{\varepsilon}{\xi}(1) \geq \overset{\varepsilon}{\xi}(x)$ for $x \in (\overset{\varepsilon}{\xi}, t)_q$, and so $1 - t \geq \overset{\varepsilon}{\xi}(1) \geq \overset{\varepsilon}{\xi}(x)$. This shows that $\langle x/t\rangle \, \overline{q} \, \overset{\varepsilon}{\xi}$, that is, $x \notin (\overset{\varepsilon}{\xi}, t)_q$, a contradiction. Thus $1 \in (\overset{\varepsilon}{\xi}, t)_q$. Let $x, y, z \in X$ be such that $x * ((y * z) * y) \in (\overset{\varepsilon}{\xi}, t)_q$ and $x \in (\overset{\varepsilon}{\xi}, t)_q$. Then $\langle (x * ((y * z) * y))/t\rangle \, q \, \overset{\varepsilon}{\xi}$ and $\langle x/t\rangle \, q \, \overset{\varepsilon}{\xi}$, that is, $\overset{\varepsilon}{\xi}(x * ((y * z) * y)) > 1 - t$ and $\overset{\varepsilon}{\xi}(x) > 1 - t$. It follows from (20) that

$$\overset{\varepsilon}{\xi}(y) \geq \min\{\overset{\varepsilon}{\xi}(x * ((y * z) * y)), \overset{\varepsilon}{\xi}(x)\} > 1 - t.$$

Hence $\langle y/t\rangle \, q \, \overset{\varepsilon}{\xi}$, and so $y \in (\overset{\varepsilon}{\xi}, t)_q$. Therefore $(\overset{\varepsilon}{\xi}, t)_q$ is a piBE-filter of $(X; *, 1)$. □

Corollary 3.15. *If* ξ *is a fpiBE-filter of* $(X; *, 1)$*, then the q-set* $(\overset{\varepsilon}{\xi}, t)_q$ *of* $\overset{\varepsilon}{\xi}$ *is a piBE-filter of* $(X; *, 1)$ *for all* $t \in (0, 1]$.

Proposition 3.16. *For the Łf-set* $\overset{\varepsilon}{\xi}$ *in* X*, if the q-set* $(\overset{\varepsilon}{\xi}, t)_q$ *of* $\overset{\varepsilon}{\xi}$ *is a piBE-filter of* $(X; *, 1)$ *for* $t \in (0, 0.5]$*, then the following arguments are satisfied.*

$$(\forall t \in (0, 0.5])\left(1 \in (\overset{\varepsilon}{\xi}, t)_{\in}\right), \tag{28}$$

$$\left(\begin{array}{l}\forall x, y, z \in X, \\ \forall t_a, t_b \in (0, 0.5]\end{array}\right)\left(\begin{array}{l}\langle (x * ((y * z) * y))/t_a\rangle \, q \, \overset{\varepsilon}{\xi}, \ \langle x/t_b\rangle \, q \, \overset{\varepsilon}{\xi} \\ \quad \Rightarrow \ y \in (\overset{\varepsilon}{\xi}, \max\{t_a, t_b\})_{\in}\end{array}\right). \tag{29}$$

Proof. Assume that the q-set $(\overset{\varepsilon}{\xi}, t)_q$ of $\overset{\varepsilon}{\xi}$ is a piBE-filter of $(X; *, 1)$ for $t \in (0, 0.5]$. Then $1 \in (\overset{\varepsilon}{\xi}, t)_q$. If $1 \notin (\overset{\varepsilon}{\xi}, t)_{\in}$ for some $t \in (0, 0.5]$, then $\langle 1/t\rangle \, \overline{\in} \, \overset{\varepsilon}{\xi}$. Hence $\overset{\varepsilon}{\xi}(1) < t \leq 1 - t$ since $t \in (0, 0.5]$, and so $\langle 1/t\rangle \, \overline{q} \, \overset{\varepsilon}{\xi}$, i.e., $1 \notin (\overset{\varepsilon}{\xi}, t)_q$. This is a conradiction, and thus $1 \in (\overset{\varepsilon}{\xi}, t)_{\in}$. Let $x, y, z \in X$ and $t_a, t_b \in (0, 0.5]$ be such that $\langle (x * ((y * z) * y))/t_a\rangle \, q \, \overset{\varepsilon}{\xi}$ and $\langle x/t_b\rangle \, q \, \overset{\varepsilon}{\xi}$. Then $x * ((y * z) * y) \in (\overset{\varepsilon}{\xi}, t_a)_q \subseteq (\overset{\varepsilon}{\xi}, \max\{t_a, t_b\})_q$ and

$$x \in (\overset{\varepsilon}{\xi}, t_b)_q \subseteq (\overset{\varepsilon}{\xi}, \max\{t_a, t_b\})_q.$$

Hence $y \in (\overset{\varepsilon}{\xi}, \max\{t_a, t_b\})_q$, and so

$$\overset{\varepsilon}{\xi}(y) > 1 - \max\{t_a, t_b\} \geq \max\{t_a, t_b\},$$

i.e., $\langle y/\max\{t_a, t_b\}\rangle \in \overset{\varepsilon}{\xi}$. Therefore $y \in (\overset{\varepsilon}{\xi}, \max\{t_a, t_b\})_{\in}$. □

Theorem 3.17. *If a Łf-set* $\overset{\varepsilon}{\xi}$ *in* X *satisfies:*

$$(\forall x \in X)(\forall t \in (0, 0.5])\left(\langle x/t\rangle \in \overset{\varepsilon}{\xi} \ \Rightarrow \ \langle 1/t\rangle \, q \, \overset{\varepsilon}{\xi}\right), \tag{30}$$

$$\left(\begin{array}{l}\forall x, y, z \in X, \\ \forall t_a, t_b \in (0, 0.5]\end{array}\right)\left(\begin{array}{l}\langle (x * ((y * z) * y))/t_a\rangle \in \overset{\varepsilon}{\xi}, \ \langle x/t_b\rangle \in \overset{\varepsilon}{\xi} \\ \quad \Rightarrow \ \langle y/\min\{t_a, t_b\}\rangle \, q \, \overset{\varepsilon}{\xi}\end{array}\right), \tag{31}$$

then the non-empty q-set $(\overset{\varepsilon}{\xi}, \min\{t_a, t_b\})_q$ *of* $\overset{\varepsilon}{\xi}$ *is a piBE-filter of* $(X; *, 1)$ *for all* $t_a, t_b \in (0, 0.5]$.

Proof. Assume that $(\overset{\varepsilon}{\xi}, \min\{t_a, t_b\})_q$ is non-empty for all $t_a, t_b \in (0, 0.5]$. Then there exists $x \in (\overset{\varepsilon}{\xi}, \min\{t_a, t_b\})_q$, and so $\overset{\varepsilon}{\xi}(x) > 1 - \min\{t_a, t_b\} \geq \min\{t_a, t_b\}$, which shows that $\langle x/\min\{t_a, t_b\}\rangle \in \overset{\varepsilon}{\xi}$. Using (30) leads to $\langle 1/\min\{t_a, t_b\}\rangle \, q \, \overset{\varepsilon}{\xi}$, i.e., $1 \in (\overset{\varepsilon}{\xi}, \min\{t_a, t_b\})_q$. Let $x, y, z \in X$ and $t_a, t_b \in (0, 0.5]$ be such that

$$x * ((y * z) * y) \in (\overset{\varepsilon}{\xi}, \min\{t_a, t_b\})_q \text{ and } x \in (\overset{\varepsilon}{\xi}, \min\{t_a, t_b\})_q.$$

Then $\overset{\varepsilon}{\xi}(x*((y*z)*y)) > 1-\min\{t_a,t_b\} \geq \min\{t_a,t_b\}$ and

$$\overset{\varepsilon}{\xi}(x) > 1-\min\{t_a,t_b\} \geq \min\{t_a,t_b\}.$$

Thus $\langle (x*((y*z)*y))/\min\{t_a,t_b\}\rangle \in \overset{\varepsilon}{\xi}$ and $\langle x/\min\{t_a,t_b\}\rangle \in \overset{\varepsilon}{\xi}$. It follows from (31) that $\langle y/\min\{t_a,t_b\}\rangle \, q \, \overset{\varepsilon}{\xi}$, i.e., $y \in (\overset{\varepsilon}{\xi}, \min\{t_a,t_b\})_q$. Therefore $(\overset{\varepsilon}{\xi}, \min\{t_a,t_b\})_q$ is a piBE-filter of $(X;*,1)$. □

Theorem 3.18. *If a Łf-set* $\overset{\varepsilon}{\xi}$ *in* X *satisfies:*

$$(\forall t \in (0.5,1])\left(1 \in (\overset{\varepsilon}{\xi}, t)_\in\right), \tag{32}$$

$$\left(\begin{array}{l} \forall x,y,z \in X, \\ \forall t_a,t_b \in (0.5,1] \end{array}\right)\left(\begin{array}{l} \langle (x*((y*z)*y))/t_a\rangle \, q \, \overset{\varepsilon}{\xi}, \ \langle x/t_b\rangle \, q \, \overset{\varepsilon}{\xi} \\ \quad \Rightarrow \ y \in (\overset{\varepsilon}{\xi}, \max\{t_a,t_b\})_\in \end{array}\right), \tag{33}$$

then the q-set $(\overset{\varepsilon}{\xi}, t)_q$ *of* $\overset{\varepsilon}{\xi}$ *is a piBE-filter of* $(X;*,1)$ *for all* $t \in (0.5,1]$.

Proof. Assume that $\overset{\varepsilon}{\xi}$ satisfies (32) and (33) for all $x,y,z \in X$ and $t,t_a,t_b \in (0.5,1]$. The condition (32) leads to $\overset{\varepsilon}{\xi}(1)+t \geq 2t > 1$, that is, $\langle 1/t\rangle \, q \, \overset{\varepsilon}{\xi}$. Hence, $1 \in (\overset{\varepsilon}{\xi}, t)_q$. Let $x,y,z \in X$ be such that $x*((y*z)*y) \in (\overset{\varepsilon}{\xi}, t)_q$ and $x \in (\overset{\varepsilon}{\xi}, t)_q$. Then $\langle (x*((y*z)*y))/t\rangle \, q \, \overset{\varepsilon}{\xi}$ and $\langle x/t\rangle \, q \, \overset{\varepsilon}{\xi}$. It follows from (33) that $y \in (\overset{\varepsilon}{\xi}, \max\{t,t\})_\in = (\overset{\varepsilon}{\xi}, t)_\in$. Hence $\overset{\varepsilon}{\xi}(y) \geq t > 1-t$ since $t > 0.5$, i.e., $y \in (\overset{\varepsilon}{\xi}, t)_q$. Consequently, $(\overset{\varepsilon}{\xi}, t)_q$ is a piBE-filter of $(X;*,1)$ for all $t \in (0.5,1]$. □

Theorem 3.19. *If* ξ *is a fpiBE-filter of* $(X;*,1)$, *then the O-set* $O(\overset{\varepsilon}{\xi})$ *of* $\overset{\varepsilon}{\xi}$ *is a piBE-filter of* $(X;*,1)$.

Proof. Let ξ be a fpiBE-filter of $(X;*,1)$. Then its Łf-set $\overset{\varepsilon}{\xi}$ is a piŁfBE-filter of $(X;*,1)$ (see Theorem 3.3). It is clear that $1 \in O(\overset{\varepsilon}{\xi})$. Let $x,y,z \in X$ be such that $x*((y*z)*y) \in O(\overset{\varepsilon}{\xi})$ and $x \in O(\overset{\varepsilon}{\xi})$. Then $\overset{\varepsilon}{\xi}(x*((y*z)*y)) > 0$ and $\overset{\varepsilon}{\xi}(x) > 0$. Note that $\langle x/t_a\rangle \in \overset{\varepsilon}{\xi}$ and $\langle (x*((y*z)*y))/t_b\rangle \in \overset{\varepsilon}{\xi}$ where $t_a := \overset{\varepsilon}{\xi}(x)$ and $t_b := \overset{\varepsilon}{\xi}(x*((y*z)*y))$. Hence $x \in (\overset{\varepsilon}{\xi}, t_a)_\in$ and $x*((y*z)*y) \in (\overset{\varepsilon}{\xi}, t_b)_\in$. It follows from (16) that $y \in (\overset{\varepsilon}{\xi}, \min\{t_a,t_b\})_\in$. Thus

$$\overset{\varepsilon}{\xi}(y) \geq \min\{t_a,t_b\} = \min\{\overset{\varepsilon}{\xi}(x), \overset{\varepsilon}{\xi}(x*((y*z)*y))\} > 0,$$

i.e., $y \in O(\overset{\varepsilon}{\xi})$. Therefore $O(\overset{\varepsilon}{\xi})$ is a piBE-filter of $(X;*,1)$. □

Theorem 3.20. *If a Łf-set* $\overset{\varepsilon}{\xi}$ *in* X *satisfies and*

$$(\forall x \in X)(\forall t \in (0,1])\left(\langle x/t\rangle \in \overset{\varepsilon}{\xi} \ \Rightarrow \ \langle 1/t\rangle \in \overset{\varepsilon}{\xi}\right), \tag{34}$$

$$\left(\begin{array}{l} \forall x,y,z \in X, \\ \forall t_a,t_b \in (0,1] \end{array}\right)\left(\begin{array}{l} \langle x/t_a\rangle \in \overset{\varepsilon}{\xi}, \ \langle (x*((y*z)*y))/t_b\rangle \in \overset{\varepsilon}{\xi} \\ \quad \Rightarrow \ \langle y/\max\{t_a,t_b\}\rangle \, q \, \overset{\varepsilon}{\xi} \end{array}\right), \tag{35}$$

then the non-empty O-set $O(\overset{\varepsilon}{\xi})$ *of* $\overset{\varepsilon}{\xi}$ *is a piBE-filter of* $(X;*,1)$.

Proof. Assume that $\overset{\varepsilon}{\xi}$ satisfies (34) and (35) and let $O(\overset{\varepsilon}{\xi})$ be the non-empty O-set of $\overset{\varepsilon}{\xi}$. Then there exists $x \in O(\overset{\varepsilon}{\xi})$, and so $t := \overset{\varepsilon}{\xi}(x) > 0$, that is, $\langle x/t\rangle \in \overset{\varepsilon}{\xi}$. Hence $\langle 1/t\rangle \in \overset{\varepsilon}{\xi}$ by (34), i.e., $\overset{\varepsilon}{\xi}(1) \geq t > 0$. Thus $1 \in O(\overset{\varepsilon}{\xi})$. Let $x,y,z \in X$ be such that $x \in O(\overset{\varepsilon}{\xi})$ and $x*((y*z)*y) \in O(\overset{\varepsilon}{\xi})$. Then $\xi(x)+\varepsilon > 1$ and $\xi(x*((y*z)*y))+\varepsilon > 1$. Note that $\langle x/t_a\rangle \in \overset{\varepsilon}{\xi}$ and $\langle (x*((y*z)*y))/t_b\rangle \in \overset{\varepsilon}{\xi}$ for $t_a := \overset{\varepsilon}{\xi}(x)$ and $t_b := \overset{\varepsilon}{\xi}(x*((y*z)*y))$. Using (35) leads to $\langle y/\max\{t_a,t_b\}\rangle \, q \, \overset{\varepsilon}{\xi}$. If $y \notin O(\overset{\varepsilon}{\xi})$, then $\overset{\varepsilon}{\xi}(y) = 0$, and so

$$\begin{aligned}
&\overset{\varepsilon}{\xi}(y) + \max\{t_a,t_b\} = \max\{t_a,t_b\} = \max\{\overset{\varepsilon}{\xi}(x), \overset{\varepsilon}{\xi}(x*((y*z)*y))\} \\
&= \max\{\max\{0, \xi(x)+\varepsilon-1\}, \max\{0, \xi(x*((y*z)*y))+\varepsilon-1\}\} \\
&= \max\{\xi(x)+\varepsilon-1, \xi(x*((y*z)*y))+\varepsilon-1\} \\
&= \max\{\xi(x), \xi(x*((y*z)*y))\}+\varepsilon-1 \\
&\leq 1+\varepsilon-1 \leq 1.
\end{aligned}$$

Hence $\langle y/\max\{t_a,t_b\}\rangle \, \overline{q} \, \overset{\varepsilon}{\xi}$, a contradiction. Thus $y \in O(\overset{\varepsilon}{\xi})$, and therefore $O(\overset{\varepsilon}{\xi})$ is a piBE-filter of $(X;*,1)$. □

Theorem 3.21. *If a Łf-set* ${}^{\varepsilon}_{\xi}$ *in* X *satisfies* $\langle 1/\varepsilon\rangle\, q\, \xi$ *and*

$$(\forall x, y, z \in X)\left(\begin{array}{l} \langle x/\varepsilon\rangle\, q\, \xi,\ \langle (x * ((y * z) * y))/\varepsilon\rangle\, q\, \xi \\ \ \Rightarrow\ \langle y/\varepsilon\rangle \in {}^{\varepsilon}_{\xi} \end{array}\right), \tag{36}$$

then the O-set $O({}^{\varepsilon}_{\xi})$ *of* ${}^{\varepsilon}_{\xi}$ *is a piBE-filter of* $(X; *, 1)$.

Proof. Let ${}^{\varepsilon}_{\xi}$ be a Łf-set in X that satisfies $\langle 1/\varepsilon\rangle\, q\, \xi$ and the condition (36). Then $\xi(1) + \varepsilon > 1$, and so ${}^{\varepsilon}_{\xi}(1) = \max\{0, \xi(1)+\varepsilon-1\} = \xi(1)+\varepsilon-1 > 0$. Thus $1 \in O({}^{\varepsilon}_{\xi})$. Let $x, y, z \in X$ be such that $x*((y*z)*y) \in O({}^{\varepsilon}_{\xi})$ and $x \in O({}^{\varepsilon}_{\xi})$. Then $\xi(x)+\varepsilon > 1$ and $\xi(x*((y*z)*y))+\varepsilon > 1$, that is, $\langle x/\varepsilon\rangle\, q\, \xi$ and $\langle (x*((y*z)*y))/\varepsilon\rangle\, q\, \xi$. It follows from (36) that $\langle y/\varepsilon\rangle \in {}^{\varepsilon}_{\xi}$. Hence ${}^{\varepsilon}_{\xi}(y) \geq \varepsilon > 0$, i.e., $y \in O({}^{\varepsilon}_{\xi})$. Therefore $O({}^{\varepsilon}_{\xi})$ is a piBE-filter of $(X; *, 1)$. □

Theorem 3.22. *Let* ${}^{\varepsilon}_{\xi}$ *be a Łf-set in* X *that satisfies:*

$$(\forall x \in X)(\forall t \in [\varepsilon, 1])\left(\langle x/t\rangle\, q\, \xi\ \Rightarrow\ \langle 1/\varepsilon\rangle \in {}^{\varepsilon}_{\xi}\right), \tag{37}$$

$$\left(\begin{array}{l} \forall x, y, z \in X, \\ \forall t_a, t_b \in [\varepsilon, 1] \end{array}\right)\left(\begin{array}{l} \langle x/t_a\rangle\, q\, \xi,\ \langle (x * ((y * z) * y))/t_b\rangle\, q\, \xi \\ \ \Rightarrow\ y \in ({}^{\varepsilon}_{\xi}, \varepsilon)_{\in} \end{array}\right). \tag{38}$$

Then the non-empty O-set $O({}^{\varepsilon}_{\xi})$ *of* ${}^{\varepsilon}_{\xi}$ *is a piBE-filter of* $(X; *, 1)$.

Proof. Suppose that ${}^{\varepsilon}_{\xi}$ satisfies (37) and (38). Let $O({}^{\varepsilon}_{\xi})$ be the non-empty O-set of ${}^{\varepsilon}_{\xi}$. Then there exists $x \in O({}^{\varepsilon}_{\xi})$, and so $\xi(x) + \varepsilon - 1 > 0$. If $t \in [\varepsilon, 1]$, then $\xi(x) + t \geq \xi(x) + \varepsilon > 1$, i.e., $\langle x/t\rangle\, q\, \xi$. Hence $\langle 1/\varepsilon\rangle \in {}^{\varepsilon}_{\xi}$ by (37), which implies that ${}^{\varepsilon}_{\xi}(1) \geq \varepsilon > 0$. Thus $1 \in O({}^{\varepsilon}_{\xi})$. Let $t_a, t_b \in [\varepsilon, 1]$ and $x, y, z \in X$ be such that $x \in O({}^{\varepsilon}_{\xi})$ and $x*((y*z)*y) \in O({}^{\varepsilon}_{\xi})$. Then $\xi(x)+t_a \geq \xi(x)+\varepsilon > 1$ and $\xi(x*((y*z)*y))+t_b \geq \xi(x)+\varepsilon > 1$, that is, $\langle x/t_a\rangle\, q\, \xi$ and $\langle (x*((y*z)*y))/t_b\rangle\, q\, \xi$. It follows from (38) that $y \in ({}^{\varepsilon}_{\xi}, \varepsilon)_{\in}$. Hence ${}^{\varepsilon}_{\xi}(y) \geq \varepsilon > 0$, and so $y \in O({}^{\varepsilon}_{\xi})$. Consequently, $O({}^{\varepsilon}_{\xi})$ is a piBE-filter of $(X; *, 1)$. □

4 Conclusions and future work

Y. B. Jun introduced the notion of Łf-sets using Łukasiewicz t-norm, and applied it to BCK/BCI-algebras. Łf-set was also applied to BE-algebras by S. S. Ahn, Y. B. Jun, E. H. Roh and S. Z. Song. In this paper, we applied the concept of Łf-sets to piBE-filters of BE-algebras, and introduced the notion of piŁfBE-filters with several properties. We discussed the relationship between fpiBE-filter and piŁfBE-filter. We derived the conditions under which ŁfBE-filter can be piŁfBE-filter, and considered characterizations of piŁfBE-filter. We explored conditions for Łf-set to be piŁfBE-filter, and found the conditions under which $\in$-set, q-set, and O-set of the Łf-set can be piBE-filter.

The ideas and results obtained in this paper will be applied to the relevant algebraic structures in the future, further examining their usability as a mathematical tool applicable to medical diagnosis systems, decision theory, and automation systems etc.

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