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Commutative ideals of BCI-algebras based on Łukasiewicz fuzzy sets

G.R. Rezaei[1] and Y.B. Jun[2]

[1] *Department of Mathematics, University of Sistan and Baluchestan, Zahedan 98131, Iran*
[2] *Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea*

grezaei@math.usb.ac.ir, skywine@gmail.com

Abstract

With the aim of applying the Łukasiewicz fuzzy set to commutative ideal in BCI-algebras, the concept of Łukasiewicz fuzzy commutative ideal is introduced, and its properties are investigated. The relationship between a Łukasiewicz fuzzy ideal and a Łukasiewicz fuzzy commutative ideal are discussed. After providing an example of a Łukasiewicz fuzzy ideal, not a Łukasiewicz fuzzy commutative ideal, conditions under which a Łukasiewicz fuzzy ideal can be a Łukasiewicz fuzzy commutative ideal are explored. Characterizations of Łukasiewicz fuzzy commutative ideals are displayed. Conditions under which $\in$-set, q-set, and O-set can be commutative ideals are found.

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1 Introduction

Ideal concepts are a very important factor in studying BCK/BCI-algebras, and studies have been conducted on various types of ideals. The commutative ideal introduced by Meng [11] in 1993 is one of these ideals. The fuzzy set acts as a bridge so that algebra theory can be applied to applied sciences. Various kinds of fuzzy sets have been used in the study of substructures such as ideals in BCK/BCI-algebras (see [4, 7, 8, 9, 10, 16]). Łukasiewicz logic, which is the logic of the Łukasiewicz t-norm, is a non-classical and many-valued logic. It was originally defined in the early 20th century by Jan Łukasiewicz as a three-valued logic. Using the idea of Łukasiewicz t-norm, Y. B. Jun [5] constructed the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and applied it to BCK-algebras and BCI-algebras. Y. B. Jun and S. Z. Song studied Łukasiewicz fuzzy (positive implicative) ideals in BCK/BCI-algebras (see [6, 15]).

For the purpose of applying the Łukasiewicz fuzzy set to a commutative ideal in BCI-algebras, we introduce the concept of Łukasiewicz fuzzy commutative ideal and study its properties. We discuss the relationship between Łukasiewicz fuzzy ideal and Łukasiewicz fuzzy commutative ideal. We give an example

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of a Łukasiewicz fuzzy ideal, not a Łukasiewicz fuzzy commutative ideal, and explore the conditions under which a Łukasiewicz fuzzy ideal can be a Łukasiewicz fuzzy commutative ideal. We discuss characterizations of Łukasiewicz fuzzy commutative ideals. We explore the conditions under which $\in$-set, q-set, and O-set can be commutative ideals.

2 Preliminaries

2.1 Basic concepts about BCK/BCI-algebras

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [2] and [3]) and was extensively investigated by several researchers. We recall the definitions and basic results required in this paper. See the books [1, 12] for further information regarding BCK-algebras and BCI-algebras.

If a set X has a special element "0" and a binary operation " $*$ " satisfying the conditions:

(I_1) $(\forall a, b, c \in X)\ (((a * b) * (a * c)) * (c * b) = 0)$,

(I_2) $(\forall a, b \in X)\ ((a * (a * b)) * b = 0)$,

(I_3) $(\forall a \in X)\ (a * a = 0)$,

(I_4) $(\forall a, b \in X)\ (a * b = 0,\, b * a = 0 \ \Rightarrow\ a = b)$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(K) $(\forall a \in X)\ (0 * a = 0)$,

then X is called a *BCK-algebra*. The BCI/BCK-algebra is written as $(X, 0)_*$.

The order relation "$\leq$" in a BCK/BCI-algebra $(X, 0)_*$ is defined as follows:

$$(\forall a, b \in X)(a \leq b \ \Leftrightarrow\ a * b = 0). \tag{1}$$

Every BCK/BCI-algebra $(X, 0)_*$ satisfies the following conditions (see [1, 12]):

$$(\forall a \in X)\,(a * 0 = a)\,, \tag{2}$$

$$(\forall a, b, c \in X)\,(a \leq b \Rightarrow a * c \leq b * c,\, c * b \leq c * a)\,, \tag{3}$$

$$(\forall a, b, c \in X)\,((a * b) * c = (a * c) * b)\,. \tag{4}$$

Every BCI-algebra $(X, 0)_*$ satisfies (see [1]):

$$(\forall a, b \in X)\,(a * (a * (a * b)) = a * b)\,, \tag{5}$$

$$(\forall a, b \in X)\,(0 * (a * b) = (0 * a) * (0 * b))\,. \tag{6}$$

A BCI-algebra $(X, 0)_*$ is said to be *commutative* (see [13]) if it satisfies:

$$(\forall a, b \in X)(a \leq b \ \Rightarrow\ a = b * (b * a)). \tag{7}$$

A subset K of a BCK/BCI-algebra $(X, 0)_*$ is called

- a *subalgebra* of X (see [1, 12]) if it satisfies:

$$(\forall a, b \in K)(a * b \in K), \tag{8}$$

- an *ideal* of X (see [1, 12]) if it satisfies:

$$0 \in K, \tag{9}$$

$$(\forall a, b \in X)(a * b \in K,\, b \in K \ \Rightarrow\ a \in K). \tag{10}$$

A subset K of a BCI-algebra $(X, 0)_*$ is called a *commutative ideal* of X (see [11]) if it satisfies (9) and

$$(\forall a, b, c \in X)\left(\begin{array}{l} (a * b) * c \in K,\, c \in K \\ \quad \Rightarrow\ a * ((b * (b * a)) * (0 * (0 * (a * b)))) \in K \end{array}\right). \tag{11}$$

2.2 Basic concepts about (Łukasiewicz) fuzzy sets

A fuzzy set ξ in a set X of the form

$$\xi(b) := \begin{cases} t \in (0,1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $\langle a/t \rangle$.

For a fuzzy set ξ in a set X, we say that a fuzzy point $\langle a/t \rangle$ is

(i) *contained* in ξ, denoted by $\langle a/t \rangle \in \xi$, (see [14]) if $\xi(a) \geq t$.

(ii) *quasi-coincident* with ξ, denoted by $\langle a/t \rangle \, q \, \xi$, (see [14]) if $\xi(a) + t > 1$.

If $\langle a/t \rangle \, \alpha \, \xi$ is not established for $\alpha \in \{\in, q\}$, it is denoted by $\langle a/t \rangle \, \overline{\alpha} \, \xi$.

A fuzzy set ξ in a BCK/BCI-algebra $(X, 0)_*$ is called

- a *fuzzy subalgebra* of $(X, 0)_*$ (see [7]) if it satisfies:

$$(\forall a, b \in X)(\xi(a * b) \geq \min\{\xi(a), \xi(b)\}). \tag{12}$$

- a *fuzzy ideal* of $(X, 0)_*$ (see [7, 16]) if it satisfies:

$$(\forall a \in X)(\xi(0) \geq \xi(a)), \tag{13}$$
$$(\forall a, b \in X)(\xi(a) \geq \min\{\xi(a * b), \xi(b)\}). \tag{14}$$

A fuzzy set ξ in a BCI-algebra $(X, 0)_*$ is called

- a *closed fuzzy ideal* of $(X, 0)_*$ (see [4]) if it is a fuzzy ideal of $(X, 0)_*$ which satisfies:

$$(\forall a \in X)(\xi(0 * a) \geq \xi(a)). \tag{15}$$

- a *fuzzy commutative ideal* of $(X, 0)_*$ (see [8]) if it satisfies (13) and

$$\xi(a * ((b * (b * a)) * (0 * (0 * (a * b))))) \geq \min\{\xi((a * b) * c), \xi(c)\} \tag{16}$$

for all $a, b, c \in X$.

Definition 2.1. [5] *Let ξ be a fuzzy set in a set X and let $\delta \in (0,1)$. A function*

$$\overset{\delta}{\xi} : X \to [0,1], \; x \mapsto \max\{0, \xi(x) + \delta - 1\}$$

is called the Łukasiewicz fuzzy set of ξ in X.

Definition 2.2. [5] *Let ξ be a fuzzy set in $(X, 0)_*$ and δ an element of $(0, 1)$. Then its Łukasiewicz fuzzy set $\overset{\delta}{\xi}$ in X is called a Łukasiewicz fuzzy subalgebra of $(X, 0)_*$ if it satisfies:*

$$\langle x/t_a \rangle \in \overset{\delta}{\xi}, \; \langle y/t_b \rangle \in \overset{\delta}{\xi} \; \Rightarrow \langle (x * y)/\min\{t_a, t_b\} \rangle \in \overset{\delta}{\xi} \tag{17}$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$.

Lemma 2.3. [5] *Let ξ be a fuzzy set in X. Then its Łukasiewicz fuzzy set $\overset{\delta}{\xi}$ in X is a Łukasiewicz fuzzy subalgebra of $(X, 0)_*$ if and only if it satisfies:*

$$(\forall x, y \in X)(\overset{\delta}{\xi}(x * y) \geq \min\{\overset{\delta}{\xi}(x), \overset{\delta}{\xi}(y)\}). \tag{18}$$

Definition 2.4. [6] *Let ξ be a fuzzy set in a BCK/BCI-algebra X. Then its Łukasiewicz fuzzy set $\overset{\delta}{\xi}$ in X is called a Łukasiewicz fuzzy ideal of X if it satisfies:*

$$\overset{\delta}{\xi}(0) \text{ is an upper bound of } \{\overset{\delta}{\xi}(x) \mid x \in X\}, \tag{19}$$
$$\langle (x * y)/t_a \rangle \in \overset{\delta}{\xi}, \; \langle y/t_b \rangle \in \overset{\delta}{\xi} \; \Rightarrow \langle x/\min\{t_a, t_b\} \rangle \in \overset{\delta}{\xi} \tag{20}$$

for all $x, y \in X$ and $t_a, t_b \in (0, 1]$.

Lemma 2.5. [6] *Let ξ be a fuzzy set in $(X,0)_*$. Then its Łukasiewicz fuzzy set ${}_{\xi}^{\delta}$ is a Łukasiewicz fuzzy ideal of $(X,0)_*$ if and only if it satisfies:*

$$(\forall x \in X)(\forall t_a \in (0,1]) \left(\langle x/t_a \rangle \in {}_{\xi}^{\delta} \;\Rightarrow\; \langle 0/t_a \rangle \in {}_{\xi}^{\delta} \right), \tag{21}$$

$$(\forall x, y \in X)({}_{\xi}^{\delta}(x) \geq \min\{{}_{\xi}^{\delta}(x * y), {}_{\xi}^{\delta}(y)\}). \tag{22}$$

Let ξ be a fuzzy set in X. For the Łukasiewicz fuzzy set ${}_{\xi}^{\delta}$ of ξ in X and $t \in (0,1]$, consider the sets

$$({}_{\xi}^{\delta}, t)_{\in} := \{x \in X \mid \langle x/t \rangle \in {}_{\xi}^{\delta}\} \text{ and } ({}_{\xi}^{\delta}, t)_q := \{x \in X \mid \langle x/t \rangle \, q \, {}_{\xi}^{\delta}\},$$

which are called the $\in$-*set* and q-*set*, respectively, of ${}_{\xi}^{\delta}$ (with value t). Also, consider a set:

$$O({}_{\xi}^{\delta}) := \{x \in X \mid {}_{\xi}^{\delta}(x) > 0\} \tag{23}$$

which is called an O-*set* of ${}_{\xi}^{\delta}$. It is observed that

$$O({}_{\xi}^{\delta}) = \{x \in X \mid \xi(x) + \delta - 1 > 0\}.$$

3 Łukasiewicz fuzzy commutative ideals in BCI-algebras

In this section, let $(X,0)_*$ be a BCI-algebra, and δ be an element of $(0,1)$ unless otherwise specified.

For any elements x and y of X, let

$$x^n * y := x * (\cdots * (x * (x * y)) \cdots),$$

where x appears n times.

Definition 3.1. *Let ξ be a fuzzy set in X. Then its Łukasiewicz fuzzy set ${}_{\xi}^{\delta}$ in X is called a Łukasiewicz fuzzy commutative ideal (briefly, ŁFC-ideal) of X if it satisfies* (19) *(or, equivalently* (21)*) and*

$$(\forall x, y, z \in X)(\forall t_a, t_c \in (0,1]) \left(\begin{array}{l} \langle ((x * y) * z)/t_a \rangle \in {}_{\xi}^{\delta},\ \langle z/t_c \rangle \in {}_{\xi}^{\delta} \\ \quad \Rightarrow \langle (x * ((y^2 * x) * (0^2 * (x * y))))/\min\{t_a, t_c\} \rangle \in {}_{\xi}^{\delta} \end{array} \right). \tag{24}$$

Example 3.2. *Let $X = \{\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ be a set with a binary operation "$*$" given as follows:*

$*$	κ_0	κ_1	κ_2	κ_3	κ_4
κ_0	κ_0	κ_0	κ_4	κ_3	κ_2
κ_1	κ_1	κ_0	κ_4	κ_3	κ_2
κ_2	κ_2	κ_2	κ_0	κ_4	κ_3
κ_3	κ_3	κ_3	κ_2	κ_0	κ_4
κ_4	κ_4	κ_4	κ_3	κ_2	κ_0

Then $(X, \kappa_0)_*$ is a BCI-algebra (see [1]). Define a fuzzy set ξ in X as follows:

$$\xi : X \to [0,1],\ x \mapsto \begin{cases} 0.97 & \text{if } x = \kappa_0, \\ 0.79 & \text{if } x = \kappa_1, \\ 0.59 & \text{if } x = \kappa_2, \\ 0.59 & \text{if } x = \kappa_3, \\ 0.59 & \text{if } x = \kappa_4. \end{cases}$$

Given $\delta := 0.58$, the Łukasiewicz fuzzy set ${}_{\xi}^{\delta}$ of ξ in X is given as follows:

$${}_{\xi}^{\delta} : X \to [0,1],\ x \mapsto \begin{cases} 0.55 & \text{if } x = \kappa_0, \\ 0.37 & \text{if } x = \kappa_1, \\ 0.17 & \text{if } x = \kappa_2, \\ 0.17 & \text{if } x = \kappa_3, \\ 0.17 & \text{if } x = \kappa_4. \end{cases}$$

It is routine to verify that ${}_{\xi}^{\delta}$ is a ŁFC ideal of $(X, \kappa_0)_*$.

Proposition 3.3. *Every ŁFC ideal* ${}^{\delta}_{\xi}$ *of* $(X,0)_*$ *satisfies:*

$$(\forall x, y \in X)(\forall t \in (0,1])\left(\langle (x*y)/t\rangle \in {}^{\delta}_{\xi} \;\Rightarrow \langle (x*((y^2*x)*(0^2*(x*y))))/t\rangle \in {}^{\delta}_{\xi}\right). \tag{25}$$

Proof. If we choose 0 instead of z, and $t := t_a = t_c$ from (24) and use (19), we will get (25). □

We discuss the relationship between Łukasiewicz fuzzy ideals and ŁFC ideals.

Theorem 3.4. *Every ŁFC ideal is a Łukasiewicz fuzzy ideal.*

Proof. Let ${}^{\delta}_{\xi}$ be a ŁFC ideal of $(X,0)_*$. Let $x, y \in X$ and $t_a, t_c \in (0,1]$ be such that $\langle (x*z)/t_a\rangle \in {}^{\delta}_{\xi}$ and $\langle z/t_c\rangle \in {}^{\delta}_{\xi}$. Then $\langle ((x*0)*z)/t_a\rangle = \langle (x*z)/t_a\rangle \in {}^{\delta}_{\xi}$, and so

$$\begin{aligned}\langle x/\min\{t_a,t_c\}\rangle &= \langle (x*0)/\min\{t_a,t_c\}\rangle \\ &= \langle (x*((0^2*x)*(0^2*(x*0)))/\min\{t_a,t_c\}\rangle \in {}^{\delta}_{\xi}\end{aligned}$$

by (I_3), (2) and (24). Hence ${}^{\delta}_{\xi}$ is a Łukasiewicz fuzzy ideal of $(X,0)_*$. □

The converse of Theorem 3.4 may not be true as shown in the following example.

Example 3.5. *Let* $X = \{\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4\}$ *be a set with a binary operation "$*$" given as follows:*

$*$	κ_0	κ_1	κ_2	κ_3	κ_4
κ_0	κ_0	κ_0	κ_0	κ_0	κ_0
κ_1	κ_1	κ_0	κ_1	κ_0	κ_0
κ_2	κ_2	κ_2	κ_0	κ_0	κ_0
κ_3	κ_3	κ_3	κ_3	κ_0	κ_0
κ_4	κ_4	κ_4	κ_4	κ_3	κ_0

Then $(X, \kappa_0)_*$ *is a BCK-algebra and so a BCI-algebra (see [1, 12]). Define a fuzzy set* ξ *in* X *as follows:*

$$\xi : X \to [0,1],\; x \mapsto \begin{cases} 0.89 & \text{if } x = \kappa_0, \\ 0.77 & \text{if } x = \kappa_1, \\ 0.43 & \text{if } x = \kappa_2, \\ 0.59 & \text{if } x = \kappa_3, \\ 0.43 & \text{if } x = \kappa_4. \end{cases}$$

Given $\delta := 0.36$*, the Łukasiewicz fuzzy set* ${}^{\delta}_{\xi}$ *of* ξ *in* X *is given as follows:*

$${}^{\delta}_{\xi} : X \to [0,1],\; x \mapsto \begin{cases} 0.25 & \text{if } x = \kappa_0, \\ 0.13 & \text{if } x = \kappa_1, \\ 0.00 & \text{if } x \in \{\kappa_2, \kappa_3, \kappa_4\}. \end{cases}$$

A simple calculation confirms that ${}^{\delta}_{\xi}$ *is a Łukasiewicz fuzzy ideal of* $(X, \kappa_0)_*$*. If we take* t_a *and* t_c *in* $(0, 0.23]$*, then* $\langle ((\kappa_2 * \kappa_3) * \kappa_0)/t_a\rangle \in {}^{\delta}_{\xi}$ *and* $\langle \kappa_0/t_c\rangle \in {}^{\delta}_{\xi}$*. But*

$$\langle (\kappa_2 * ((\kappa_3^2 * \kappa_2) * (\kappa_0^2 * (\kappa_2 * \kappa_3))))/\min\{t_a,t_c\}\rangle = \langle \kappa_2/\min\{t_a,t_c\}\rangle \,\overline{\in}\, {}^{\delta}_{\xi}.$$

Hence ${}^{\delta}_{\xi}$ *is not a ŁFC ideal of* $(X, \kappa_0)_*$*.*

We explore the conditions under which a Łukasiewicz fuzzy ideal becomes ŁFC ideal.

Theorem 3.6. *If a Łukasiewicz fuzzy ideal* ${}^{\delta}_{\xi}$ *of* $(X,0)_*$ *satisfies the condition* (25)*, then it is a ŁFC ideal of* $(X,0)_*$*.*

Proof. Assume that a Łukasiewicz fuzzy ideal ${}^{\delta}_{\xi}$ of $(X,0)_*$ satisfies the condition (25). It is clear that ${}^{\delta}_{\xi}$ satisfies the condition (19). Since $\langle (x*y)/{}^{\delta}_{\xi}(x*y)\rangle \in {}^{\delta}_{\xi}$ for all $x,y\in X$, it follows from (25) that

$$\langle (x*((y^2*x)*(0^2*(x*y))))/{}^{\delta}_{\xi}(x*y)\rangle \in {}^{\delta}_{\xi}.$$

Hence ${}^{\delta}_{\xi}(x*((y^2*x)*(0^2*(x*y)))) \geq {}^{\delta}_{\xi}(x*y)$ for all $x,y\in X$. Let $x,y,z\in X$ and $t_a,t_c\in(0,1]$ be such that $\langle ((x*y)*z)/t_a\rangle \in {}^{\delta}_{\xi}$ and $\langle z/t_c\rangle \in {}^{\delta}_{\xi}$. Then $\langle (x*y)/\min\{t_a,t_c\}\rangle \in {}^{\delta}_{\xi}$ by (20), and so

$${}^{\delta}_{\xi}(x*((y^2*x)*(0^2*(x*y)))) \geq {}^{\delta}_{\xi}(x*y) \geq \min\{t_a,t_c\},$$

that is, $\langle (x*((y^2*x)*(0^2*(x*y))))/\min\{t_a,t_c\}\rangle \in {}^{\delta}_{\xi}$. Therefore, ${}^{\delta}_{\xi}$ is a ŁFC ideal of $(X,0)_*$. □

Definition 3.7. [6] *A Łukasiewicz fuzzy ideal ${}^{\delta}_{\xi}$ of $(X,0)_*$ is said to be closed if it is also a Łukasiewicz fuzzy subalgebra of $(X,0)_*$.*

Theorem 3.8. *Every Łukasiewicz fuzzy ideal ${}^{\delta}_{\xi}$ of $(X,0)_*$ is closed if and only if it satisfies:*

$$(\forall x\in X)(\forall t\in(0,1])\left(\langle x/t\rangle \in {}^{\delta}_{\xi} \ \Rightarrow\ \langle (0*x)/t\rangle \in {}^{\delta}_{\xi}\right). \tag{26}$$

Proof. Assume that a Łukasiewicz fuzzy ideal ${}^{\delta}_{\xi}$ of $(X,0)_*$ is closed. Let $x\in X$ and $t\in(0,1]$ be such that $\langle x/t\rangle \in {}^{\delta}_{\xi}$. Then $\langle 0/t\rangle \in {}^{\delta}_{\xi}$ by (21), and so $\langle (0*x)/t\rangle = \langle (0*x)/\min\{t,t\}\rangle \in {}^{\delta}_{\xi}$ by (17).

Conversely, let ${}^{\delta}_{\xi}$ be a Łukasiewicz fuzzy ideal of $(X,0)_*$ that satisfies (26). Let $x,y\in X$ and $t_a,t_b\in(0,1]$ be such that $\langle x/t_a\rangle \in {}^{\delta}_{\xi}$ and $\langle y/t_b\rangle \in {}^{\delta}_{\xi}$. Then $\langle ((x*y)*x)/t_b\rangle = \langle (0*y)/t_b\rangle \in {}^{\delta}_{\xi}$ by (I_3), (4) and (26). It follows from (20) that $\langle (x*y)/\min\{t_a,t_b\}\rangle \in {}^{\delta}_{\xi}$. Consequently, ${}^{\delta}_{\xi}$ is a closed Łukasiewicz fuzzy ideal of $(X,0)_*$. □

Lemma 3.9. [6] *Every Łukasiewicz fuzzy ideal ${}^{\delta}_{\xi}$ of X satisfies:*

$$(\forall x,y,z\in X)(\forall t_b,t_c\in(0,1])\begin{pmatrix} x*y\leq z,\ \langle y/t_b\rangle \in {}^{\delta}_{\xi},\ \langle z/t_c\rangle \in {}^{\delta}_{\xi} \\ \Rightarrow\ \langle x/\min\{t_b,t_c\}\rangle \in {}^{\delta}_{\xi}\end{pmatrix}, \tag{27}$$

which is equivalent to the following assertion.

$$\left(\forall x,y,z\in X)(x*y\leq z\ \Rightarrow\ {}^{\delta}_{\xi}(x)\geq \min\{{}^{\delta}_{\xi}(y),{}^{\delta}_{\xi}(z)\}\right). \tag{28}$$

Theorem 3.10. *Let ${}^{\delta}_{\xi}$ be a closed Łukasiewicz fuzzy ideal of $(X,0)_*$. Then it is a ŁFC ideal of $(X,0)_*$ if and only if it satisfies:*

$$(\forall x,y\in X)(\forall t\in(0,1])\left(\langle (x*y)/t\rangle \in {}^{\delta}_{\xi}\ \Rightarrow\ \langle (x*(y^2*x))/t\rangle \in {}^{\delta}_{\xi}\right). \tag{29}$$

Proof. Let ${}^{\delta}_{\xi}$ be a closed Łukasiewicz fuzzy ideal of $(X,0)_*$. Assume that ${}^{\delta}_{\xi}$ is a ŁFC ideal of $(X,0)_*$. Let $x,y\in X$ and $t\in(0,1]$ be such that $\langle (x*y)/t\rangle \in {}^{\delta}_{\xi}$. Since $\langle (x*y)/{}^{\delta}_{\xi}(x*y)\rangle \in {}^{\delta}_{\xi}$, we have $\langle (x*((y^2*x)*(0^2*(x*y))))/{}^{\delta}_{\xi}(x*y)\rangle \in {}^{\delta}_{\xi}$ by Proposition 3.3, that is,

$${}^{\delta}_{\xi}(x*((y^2*x)*(0^2*(x*y)))) \geq {}^{\delta}_{\xi}(x*y).$$

Since

$$\begin{aligned}
&(x*(y^2*x))*(x*((y^2*x)*(0^2*(x*y))))\\
&\leq ((y^2*x)*(0^2*(x*y)))*(y^2*x)\\
&= ((y^2*x)*(y^2*x))*(0^2*(x*y))\\
&= 0*(0^2*(x*y)) = 0*(x*y),
\end{aligned}$$

it follows from Theorem 3.8 and Lemma 3.9 that

$$\begin{aligned}\overset{\delta}{\xi}(x*(y^2*x)) &\geq \min\{\overset{\delta}{\xi}(x*((y^2*x)*(0^2*(x*y)))), \overset{\delta}{\xi}(0*(x*y))\}\\ &\geq \min\{\overset{\delta}{\xi}(x*y), \overset{\delta}{\xi}(0*(x*y))\}\\ &= \overset{\delta}{\xi}(x*y) \geq t,\end{aligned}$$

i.e., $\langle (x*(y^2*x))/t\rangle \in \overset{\delta}{\xi}$.

Conversely, let $\overset{\delta}{\xi}$ be a closed Łukasiewicz fuzzy ideal of $(X,0)_*$ satisfying the condition (29). Let $x, y \in X$ and $t \in (0,1]$ be such that $\langle (x*y)/t\rangle \in \overset{\delta}{\xi}$. Then $\overset{\delta}{\xi}(x*y) \geq t$. Since $\langle (x*y)/\overset{\delta}{\xi}(x*y)\rangle \in \overset{\delta}{\xi}$, we get $\langle (x*(y^2*x))/\overset{\delta}{\xi}(x*y)\rangle \in \overset{\delta}{\xi}$ by (29), and so $\overset{\delta}{\xi}(x*(y^2*x)) \geq \overset{\delta}{\xi}(x*y)$. Since

$$\begin{aligned}&(x*((y^2*x)*(0^2*(x*y))))*(x*(y^2*x))\\ &\leq (y^2*x)*((y^2*x)*(0^2*(x*y)))\\ &\leq 0^2*(x*y),\end{aligned}$$

we have

$$\begin{aligned}&\overset{\delta}{\xi}(x*((y^2*x)*(0^2*(x*y)))) \geq \min\{\overset{\delta}{\xi}(x*(y^2*x)), \overset{\delta}{\xi}(0^2*(x*y))\}\\ &\geq \min\{\overset{\delta}{\xi}(x*y), \overset{\delta}{\xi}(0^2*(x*y))\} = \overset{\delta}{\xi}(x*y) \geq t\end{aligned}$$

by Theorem 3.8 and Lemma 3.9. Hence $\langle (x*((y^2*x)*(0^2*(x*y))))/t\rangle \in \overset{\delta}{\xi}$, and therefore $\overset{\delta}{\xi}$ is a ŁFC ideal of $(X,0)_*$ by Theorem 3.6. □

Lemma 3.11. [13] *A BCI-algebra is commutative if and only if it satisfies:*

$$(\forall x, y \in X)(x^2*y = y^2*(x^2*y)). \tag{30}$$

Theorem 3.12. *In a commutative BCI-algebra, every closed Łukasiewicz fuzzy ideal is a ŁFC ideal.*

Proof. Let $\overset{\delta}{\xi}$ be a closed Łukasiewicz fuzzy ideal of a commutative BCI-algebra $(X,0)_*$. Let $x, y \in X$ and $t \in (0,1]$ be such that $\langle (x*y)/t\rangle \in \overset{\delta}{\xi}$. Using (I_1), (I_3), (4), (5), and Lemma 3.11 leads to

$$\begin{aligned}&(x*(y^2*x))*(x*y) = (x^2*y)*(y^2*x) = (y^2*(x^2*y))*(y^2*x)\\ &= (y^3*x)*(y*(x^2*y)) = (y*x)*(y*(x^2*y)) \leq (x^2*y)*x = 0*(x*y).\end{aligned}$$

It follows from Theorem 3.8 and Lemma 3.9 that

$$\overset{\delta}{\xi}(x*(y^2*x)) \geq \min\{\overset{\delta}{\xi}(x*y), \overset{\delta}{\xi}(0*(x*y))\} = \overset{\delta}{\xi}(x*y) \geq t,$$

that is, $\langle (x*(y^2*x))/t\rangle \in \overset{\delta}{\xi}$. Therefore, $\overset{\delta}{\xi}$ is a ŁFC ideal of $(X,0)_*$ by Theorem 3.10. □

The theorem below reveals that an ŁFC ideal can be derived from fuzzy commutative ideal.

Theorem 3.13. *If ξ is a fuzzy commutative ideal of $(X,0)_*$, then its Łukasiewicz fuzzy set $\overset{\delta}{\xi}$ is a ŁFC ideal of $(X,0)_*$.*

Proof. Let ξ be a fuzzy commutative ideal of $(X,0)_*$. Then

$$\overset{\delta}{\xi}(0) = \max\{0, \xi(0)+\delta-1\} \geq \max\{0, \xi(x)+\delta-1\} = \overset{\delta}{\xi}(x)$$

for all $x \in X$. Hence $\overset{\delta}{\xi}(0)$ is an upper bound of $\{\overset{\delta}{\xi}(x) \mid x \in X\}$. Let $x, y, z \in X$ and $t_a, t_c \in (0,1]$ be such that $\langle ((x*y)*z)/t_a\rangle \in \overset{\delta}{\xi}$ and $\langle z/t_c\rangle \in \overset{\delta}{\xi}$. Then $\overset{\delta}{\xi}((x*y)*z) \geq t_a$ and $\overset{\delta}{\xi}(z) \geq t_c$, which imply that

$$\begin{aligned}\overset{\delta}{\xi}(x*((y^2*x)*(0^2*(x*y)))) &= \max\{0, \xi(x*((y^2*x)*(0^2*(x*y))))+\delta-1\}\\ &\geq \max\{0, \min\{\xi((x*y)*z), \xi(z)\}+\delta-1\}\\ &= \max\{0, \min\{\xi((x*y)*z)+\delta-1, \xi(z)+\delta-1\}\}\\ &= \min\{\max\{0, \xi((x*y)*z)+\delta-1\}, \max\{0, \xi(z)+\delta-1\}\}\\ &= \min\{\overset{\delta}{\xi}((x*y)*z), \overset{\delta}{\xi}(z)\} \geq \min\{t_a, t_c\}.\end{aligned}$$

Hence $\langle (x*((y^2*x)*(0^2*(x*y))))/\min\{t_a, t_c\}\rangle \in \overset{\delta}{\xi}$, and therefore $\overset{\delta}{\xi}$ is a ŁFC ideal of X. □

We explore the conditions under which $\in$-set, q-set, and O-set can be commutative ideals.

Theorem 3.14. *Let* $\overset{\delta}{\xi}$ *be the Łukasiewicz fuzzy set of a fuzzy set* ξ *in* X. *Then the* $\in$*-set* $(\overset{\delta}{\xi}, t)_\in$ *of* $\overset{\delta}{\xi}$ *is a commutative ideal of* $(X, 0)_*$ *for all* $t \in (0.5, 1]$ *if and only if the following assertions are valid.*

$$(\forall x \in X)\left(\overset{\delta}{\xi}(x) \leq \max\{\overset{\delta}{\xi}(0), 0.5\}\right), \tag{31}$$

$$(\forall x, y, z \in X)\left(\min\{\overset{\delta}{\xi}((x * y) * z), \overset{\delta}{\xi}(z)\} \leq \max\{\overset{\delta}{\xi}(x * ((y^2 * x) * (0^2 * (x * y)))), 0.5\}\right). \tag{32}$$

Proof. Assume that $(\overset{\delta}{\xi}, t)_\in$ is a commutative ideal of $(X, 0)_*$ for $t \in (0.5, 1]$. If

$$\overset{\delta}{\xi}(a) > \max\{\overset{\delta}{\xi}(0), 0.5\},$$

for some $a \in X$, then $\overset{\delta}{\xi}(a) \in (0.5, 1]$ and $\overset{\delta}{\xi}(a) > \overset{\delta}{\xi}(0)$. If we take $t := \overset{\delta}{\xi}(a)$, then $\langle a/t \rangle \in \overset{\delta}{\xi}$, that is, $a \in (\overset{\delta}{\xi}, t)_\in$, and $0 \notin (\overset{\delta}{\xi}, t)_\in$. This is a contradiction, and so $\overset{\delta}{\xi}(x) \leq \max\{\overset{\delta}{\xi}(0), 0.5\}$ for all $x \in X$. Now, suppose that the condition (32) is not valid. Then there exist $x, y, z \in X$ such that

$$\min\{\overset{\delta}{\xi}((x * y) * z), \overset{\delta}{\xi}(z)\} > \max\{\overset{\delta}{\xi}(x * ((y^2 * x) * (0^2 * (x * y)))), 0.5\}.$$

If we take $t := \min\{\overset{\delta}{\xi}((x*y)*z), \overset{\delta}{\xi}(z)\}$, then $t \in (0.5, 1]$ and $\langle ((x*y)*z)/t \rangle$, $\langle z/t \rangle \in (\overset{\delta}{\xi}, t)_\in$, i.e., $(x*y)*z$, $z \in (\overset{\delta}{\xi}, t)_\in$. Since $(\overset{\delta}{\xi}, t)_\in$ is a commutative ideal of X, we have $x * ((y^2 * x) * (0^2 * (x * y))) \in (\overset{\delta}{\xi}, t)_\in$. But $\overset{\delta}{\xi}(x * ((y^2 * x) * (0^2 * (x * y)))) < t$ implies $x * ((y^2 * x) * (0^2 * (x * y))) \notin (\overset{\delta}{\xi}, t)_\in$, a contradiction. Hence the condition (32) is valid.

Conversely, suppose that $\overset{\delta}{\xi}$ satisfies (31) and (32). Let $t \in (0.5, 1]$. For every $x \in (\overset{\delta}{\xi}, t)_\in$, we have $0.5 < t \leq \overset{\delta}{\xi}(x) \leq \max\{\overset{\delta}{\xi}(0), 0.5\}$ by (31). Thus $0 \in (\overset{\delta}{\xi}, t)_\in$. Let $x, y, z \in X$ be such that $(x * y) * z \in (\overset{\delta}{\xi}, t)_\in$ and $z \in (\overset{\delta}{\xi}, t)_\in$. Then $\overset{\delta}{\xi}((x * y) * z) \geq t$ and $\overset{\delta}{\xi}(z) \geq t$, which imply from (32) that

$$0.5 < t \leq \min\{\overset{\delta}{\xi}((x * y) * z), \overset{\delta}{\xi}(z)\} \leq \max\{\overset{\delta}{\xi}(x * ((y^2 * x) * (0^2 * (x * y)))), 0.5\}.$$

Hense $\langle (x * ((y^2 * x) * (0^2 * (x * y))))/t \rangle \in \overset{\delta}{\xi}$, i.e., $x * ((y^2 * x) * (0^2 * (x * y))) \in (\overset{\delta}{\xi}, t)_\in$. Therefore $(\overset{\delta}{\xi}, t)_\in$ is a commutative ideal of X for $t \in (0.5, 1]$. □

Theorem 3.15. *If a Łukasiewicz fuzzy set* $\overset{\delta}{\xi}$ *in* X *satisfies:*

$$(\forall x \in X)(\forall t \in (0.5, 1])\left(\langle x/t \rangle \, q \, \overset{\delta}{\xi} \;\Rightarrow\; \langle 0/t \rangle \in \overset{\delta}{\xi}\right), \tag{33}$$

$$(\forall x, y, z \in X)(\forall t_a, t_c \in (0.5, 1])\left(\begin{array}{l} \langle ((x * y) * z)/t_a \rangle \, q \, \overset{\delta}{\xi},\ \langle z/t_c \rangle \, q \, \overset{\delta}{\xi} \\ \quad \Rightarrow\ \langle (x * ((y^2 * x) * (0^2 * (x * y))))/\max\{t_a, t_c\} \rangle \in \overset{\delta}{\xi} \end{array}\right), \tag{34}$$

then the non-empty $\in$*-set* $(\overset{\delta}{\xi}, \max\{t_a, t_c\})_\in$ *of* $\overset{\delta}{\xi}$ *is a commutative ideal of* $(X, 0)_*$ *for all* $t_a, t_c \in (0.5, 1]$.

Proof. Let $t_a, t_c \in (0.5, 1]$ and assume that the $\in$-set $(\overset{\delta}{\xi}, \max\{t_a, t_c\})_\in$ of $\overset{\delta}{\xi}$ is non-empty. Then there exists $x \in (\overset{\delta}{\xi}, \max\{t_a, t_c\})_\in$, and so $\overset{\delta}{\xi}(x) \geq \max\{t_a, t_c\} > 1 - \max\{t_a, t_c\}$, i.e., $\langle x/\max\{t_a, t_c\} \rangle \, q \, \overset{\delta}{\xi}$. Hence $\langle 0/\max\{t_a, t_c\} \rangle \in \overset{\delta}{\xi}$ by (33), and thus $0 \in (\overset{\delta}{\xi}, \max\{t_a, t_c\})_\in$. Let $x, y, z \in X$ be such that $(x * y) * z \in (\overset{\delta}{\xi}, \max\{t_a, t_c\})_\in$ and $z \in (\overset{\delta}{\xi}, \max\{t_a, t_c\})_\in$. Then $\overset{\delta}{\xi}((x * y) * z) \geq \max\{t_a, t_c\} > 1 - \max\{t_a, t_c\}$ and $\overset{\delta}{\xi}(z) \geq \max\{t_a, t_c\} > 1 - \max\{t_a, t_c\}$, that is, $\langle ((x*y)*z)/\max\{t_a, t_c\} \rangle \, q \, \overset{\delta}{\xi}$ and $\langle z/\max\{t_a, t_c\} \rangle \, q \, \overset{\delta}{\xi}$. It follows from (34) that

$$\langle (x * ((y^2 * x) * (0^2 * (x * y))))/\max\{t_a, t_c\} \rangle \in \overset{\delta}{\xi}.$$

Hence $x * ((y^2 * x) * (0^2 * (x * y))) \in (\overset{\delta}{\xi}, \max\{t_a, t_c\})_\in$, and therefore $(\overset{\delta}{\xi}, \max\{t_a, t_c\})_\in$ is a commutative ideal of $(X, 0)_*$ for all $t_a, t_c \in (0.5, 1]$. □

Theorem 3.16. *If a Łukasiewicz fuzzy set* $\overset{\delta}{\xi}$ *in* X *satisfies* (33) *and*

$$(\forall x, y, z \in X)(\forall t_a, t_c \in (0.5, 1])\left(\begin{array}{l} \langle ((x * y) * z)/t_a \rangle \, q \, \overset{\delta}{\xi},\ \langle z/t_c \rangle \, q \, \overset{\delta}{\xi} \\ \quad \Rightarrow\ \langle (x * ((y^2 * x) * (0^2 * (x * y))))/\min\{t_a, t_c\} \rangle \in \overset{\delta}{\xi} \end{array}\right), \tag{35}$$

then the non-empty $\in$*-set* $(\overset{\delta}{\xi}, \min\{t_a, t_c\})_\in$ *of* $\overset{\delta}{\xi}$ *is a commutative ideal of* $(X, 0)_*$ *for all* $t_a, t_c \in (0.5, 1]$.

Proof. It can be verified through a process similar to the proof in Theorem 3.15. □

Lemma 3.17. *Every ŁFC ideal* ${}^{\delta}_{\xi}$ *of* $(X,0)_*$ *satisfies:*

$$(\forall x,y,z\in X)({}^{\delta}_{\xi}(x*((y^2*x)*(0^2*(x*y))))\geq \min\{{}^{\delta}_{\xi}((x*y)*z),{}^{\delta}_{\xi}(z)\}). \tag{36}$$

Proof. Note that $\langle((x*y)*z)/{}^{\delta}_{\xi}((x*y)*z)\rangle\in{}^{\delta}_{\xi}$ and $\langle z/{}^{\delta}_{\xi}(z)\rangle\in{}^{\delta}_{\xi}$ for all $x,y,z\in X$. It follows from (24) that

$$\langle(x*((y^2*x)*(0^2*(x*y))))/\min\{{}^{\delta}_{\xi}((x*y)*z),{}^{\delta}_{\xi}(z)\}\rangle\in{}^{\delta}_{\xi}.$$

Hence ${}^{\delta}_{\xi}(x*((y^2*x)*(0^2*(x*y))))\geq\min\{{}^{\delta}_{\xi}((x*y)*z),{}^{\delta}_{\xi}(z)\}$ for all $x,y,z\in X$. □

Theorem 3.18. *If the Łukasiewicz fuzzy set* ${}^{\delta}_{\xi}$ *of a fuzzy set* ξ *in* X *is a ŁFC ideal of* X*, then its q-set* $({}^{\delta}_{\xi},t)_q$ *is a commutative ideal of* X *for all* $t\in(0,1]$.

Proof. Assume that ${}^{\delta}_{\xi}$ is a ŁFC ideal of $(X,0)_*$ and let $t\in(0,1]$. If $0\notin({}^{\delta}_{\xi},t)_q$, then $\langle 0/t\rangle\,\overline{q}\,{}^{\delta}_{\xi}$, that is, ${}^{\delta}_{\xi}(0)+t\leq 1$. Since ${}^{\delta}_{\xi}(0)\geq{}^{\delta}_{\xi}(x)$ for $x\in({}^{\delta}_{\xi},t)_q$, it follows that ${}^{\delta}_{\xi}(x)\leq{}^{\delta}_{\xi}(0)\leq 1-t$. Hence $\langle x/t\rangle\,\overline{q}\,{}^{\delta}_{\xi}$, and so $x\notin({}^{\delta}_{\xi},t)_q$. This is a contradiction, and so $0\in({}^{\delta}_{\xi},t)_q$. Let $x,y,z\in X$ be such that $(x*y)*z\in({}^{\delta}_{\xi},t)_q$ and $z\in({}^{\delta}_{\xi},t)_q$. Then $\langle((x*y)*z)/t\rangle\,q\,{}^{\delta}_{\xi}$ and $\langle z/t\rangle\,q\,{}^{\delta}_{\xi}$, that is, ${}^{\delta}_{\xi}((x*y)*z)>1-t$ and ${}^{\delta}_{\xi}(z)>1-t$. It follows from Lemma 3.17 that

$${}^{\delta}_{\xi}(x*((y^2*x)*(0^2*(x*y))))\geq\min\{{}^{\delta}_{\xi}((x*y)*z),{}^{\delta}_{\xi}(z)\}>1-t.$$

Thus $\langle(x*((y^2*x)*(0^2*(x*y))))/t\rangle\,q\,{}^{\delta}_{\xi}$ and so $x*((y^2*x)*(0^2*(x*y)))\in({}^{\delta}_{\xi},t)_q$. Therefore $({}^{\delta}_{\xi},t)_q$ is a commutative ideal of $(X,0)_*$. □

Corollary 3.19. *If* ξ *is a fuzzy commutative ideal of* $(X,0)_*$*, then the q-set* $({}^{\delta}_{\xi},t)_q$ *of* ${}^{\delta}_{\xi}$ *is a commutative ideal of* X *for all* $t\in(0,1]$.

Theorem 3.20. *Let* ξ *be a fuzzy set in* X*. For the Łukasiewicz fuzzy set* ${}^{\delta}_{\xi}$ *of* ξ *in* X*, if the q-set* $({}^{\delta}_{\xi},t)_q$ *of* ${}^{\delta}_{\xi}$ *is a commutative ideal of* X*, then the following assertions are valid.*

$$0\in({}^{\delta}_{\xi},t_a)_{\in}, \tag{37}$$

$$\langle((x*y)*z)/t_a\rangle\,q\,{}^{\delta}_{\xi},\ \langle z/t_c\rangle\,q\,{}^{\delta}_{\xi}\ \Rightarrow\ x*((y^2*x)*(0^2*(x*y)))\in({}^{\delta}_{\xi},\max\{t_a,t_b\})_{\in} \tag{38}$$

for all $x,y\in X$ *and* $t_a,t_c\in(0,0.5]$.

Proof. Let $x,y\in X$ and $t_a,t_c\in(0,0.5]$. If $0\notin({}^{\delta}_{\xi},t_a)_{\in}$, then $\langle 0/t_a\rangle\,\overline{\in}\,{}^{\delta}_{\xi}$ and so ${}^{\delta}_{\xi}(0)<t_a\leq 1-t_a$ since $t_a\leq 0.5$. Hence $\langle 0/t_a\rangle\,\overline{q}\,{}^{\delta}_{\xi}$ and thus $0\notin({}^{\delta}_{\xi},t_a)_q$. This is a contradiction, and therefore $0\in({}^{\delta}_{\xi},t_a)_{\in}$. Let $\langle((x*y)*z)/t_a\rangle\,q\,{}^{\delta}_{\xi}$ and $\langle z/t_c\rangle\,q\,{}^{\delta}_{\xi}$. Then $(x*y)*z\in({}^{\delta}_{\xi},t_a)_q\subseteq({}^{\delta}_{\xi},\max\{t_a,t_c\})_q$ and $z\in({}^{\delta}_{\xi},t_c)_q\subseteq({}^{\delta}_{\xi},\max\{t_a,t_c\})_q$. Hence $x*((y^2*x)*(0^2*(x*y)))\in({}^{\delta}_{\xi},\max\{t_a,t_c\})_q$, and so

$${}^{\delta}_{\xi}(x*((y^2*x)*(0^2*(x*y))))>1-\max\{t_a,t_c\}\geq\max\{t_a,t_c\},$$

that is, $\langle(x*((y^2*x)*(0^2*(x*y))))/\max\{t_a,t_c\}\rangle\in{}^{\delta}_{\xi}$. Therefore $x*((y^2*x)*(0^2*(x*y)))\in({}^{\delta}_{\xi},\max\{t_a,t_c\})_{\in}$. □

Theorem 3.21. *If a Łukasiewicz fuzzy set* ${}^{\delta}_{\xi}$ *in* X *satisfies:*

$$(\forall x\in X)(\forall t\in(0,0.5])\left(\langle x/t\rangle\in{}^{\delta}_{\xi}\ \Rightarrow\ \langle 0/t\rangle\,q\,{}^{\delta}_{\xi}\right), \tag{39}$$

and

$$(\forall x,y,z\in X)(\forall t_a,t_c\in(0,0.5])\left(\begin{array}{l}\langle((x*y)*z)/t_a\rangle\in{}^{\delta}_{\xi},\ \langle z/t_c\rangle\in{}^{\delta}_{\xi}\\ \quad\Rightarrow\ \langle(x*((y^2*x)*(0^2*(x*y))))/\min\{t_a,t_c\}\rangle\,q\,{}^{\delta}_{\xi}\end{array}\right), \tag{40}$$

then the non-empty q-set $({}^{\delta}_{\xi},\min\{t_a,t_c\})_q$ *of* ${}^{\delta}_{\xi}$ *is a commutative ideal of* $(X,0)_*$ *for all* $t_a,t_c\in(0,0.5]$.

Proof. Let $t_a, t_c \in (0, 0.5]$. If $({}^{\delta}_{\xi}, \min\{t_a, t_c\})_q$ is non-empty, then there exists $x \in ({}^{\delta}_{\xi}, \min\{t_a, t_c\})_q$. Hence ${}^{\delta}_{\xi}(x) > 1 - \min\{t_a, t_c\} \geq \min\{t_a, t_c\}$, which shows that $\langle x/\min\{t_a, t_c\}\rangle \in {}^{\delta}_{\xi}$. It follows from (39) that $\langle 0/\min\{t_a, t_c\}\rangle \, q \, {}^{\delta}_{\xi}$. Thus $0 \in ({}^{\delta}_{\xi}, \min\{t_a, t_c\})_q$. Let $x, y, z \in X$ be such that $(x * y) * z \in ({}^{\delta}_{\xi}, \min\{t_a, t_c\})_q$ and $z \in ({}^{\delta}_{\xi}, \min\{t_a, t_c\})_q$. Then ${}^{\delta}_{\xi}((x * y) * z) > 1 - \min\{t_a, t_c\} \geq \min\{t_a, t_c\}$ and ${}^{\delta}_{\xi}(z) > 1 - \min\{t_a, t_c\} \geq \min\{t_a, t_c\}$. Thus $\langle ((x * y) * z)/\min\{t_a, t_c\}\rangle \in {}^{\delta}_{\xi}$ and $\langle z/\min\{t_a, t_c\}\rangle \in {}^{\delta}_{\xi}$. It follows from (40) that $\langle (x * ((y^2 * x) * (0^2 * (x * y))))/\min\{t_a, t_c\}\rangle \, q \, {}^{\delta}_{\xi}$, i.e., $x * ((y^2 * x) * (0^2 * (x * y))) \in ({}^{\delta}_{\xi}, \min\{t_a, t_c\})_q$. Therefore $({}^{\delta}_{\xi}, \min\{t_a, t_c\})_q$ is a commutative ideal of $(X, 0)_*$. □

Theorem 3.22. *If a Łukasiewicz fuzzy set* ${}^{\delta}_{\xi}$ *in* X *satisfies* (37) *and* (38) *for all* $x, y, z \in X$ *and* $t_a, t_c \in (0.5, 1]$, *then the q-set* $({}^{\delta}_{\xi}, t)_q$ *of* ${}^{\delta}_{\xi}$ *is a commutative ideal of* $(X, 0)_*$ *for all* $t \in (0.5, 1]$.

Proof. Let $t \in (0.5, 1]$. Assume that ${}^{\delta}_{\xi}$ satisfies (37) and (38) for all $x, y, z \in X$. The condition (37) induces ${}_{\xi}(0) + t \geq 2t > 1$, i.e., $\langle 0/t\rangle \, q \, {}^{\delta}_{\xi}$. Hence $0 \in ({}^{\delta}_{\xi}, t)_q$. Let $x, y, z \in X$ be such that $(x*y)*z \in ({}^{\delta}_{\xi}, t)_q$ and $z \in ({}^{\delta}_{\xi}, t)_q$. Then $\langle ((x*y)*z)/t\rangle \, q \, {}^{\delta}_{\xi}$ and $\langle z/t\rangle \, q \, {}^{\delta}_{\xi}$. It follows from (38) that $x*((y^2*x)*(0^2*(x*y))) \in ({}^{\delta}_{\xi}, \max\{t, t\})_\in = ({}^{\delta}_{\xi}, t)_\in$. Hence ${}^{\delta}_{\xi}(x * ((y^2 * x) * (0^2 * (x * y)))) \geq t > 1 - t$, that is, $x * ((y^2 * x) * (0^2 * (x * y))) \in ({}^{\delta}_{\xi}, t)_q$. Therefore $({}^{\delta}_{\xi}, t)_q$ is a commutative ideal of $(X, 0)_*$. □

Theorem 3.23. *If* ξ *is a fuzzy commutative ideal of* $(X, 0)_*$, *then the non-empty O-set of* ${}^{\delta}_{\xi}$ *is a commutative ideal of* $(X, 0)_*$.

Proof. If ξ is a fuzzy commutative ideal of $(X, 0)_*$, then ${}^{\delta}_{\xi}$ is a ŁFC ideal of $(X, 0)_*$ (see Theorem 3.13). It is clear that $0 \in O({}^{\delta}_{\xi})$. Let $x, y, z \in X$ be such that $z \in O({}^{\delta}_{\xi})$ and $(x * y) * z \in O({}^{\delta}_{\xi})$. Then ${}^{\delta}_{\xi}((x * y) * z) > 0$ and ${}^{\delta}_{\xi}(z) > 0$. Since $\langle ((x * y) * z)/{}^{\delta}_{\xi}((x * y) * z)\rangle \in {}^{\delta}_{\xi}$ and $\langle z/{}^{\delta}_{\xi}(z)\rangle \in {}^{\delta}_{\xi}$, we have

$$\langle (x * ((y^2 * x) * (0^2 * (x * y))))/\min\left\{{}^{\delta}_{\xi}((x * y) * z), {}^{\delta}_{\xi}(z)\right\}\rangle \in {}^{\delta}_{\xi}$$

by (24). It follows that

$${}^{\delta}_{\xi}(x * ((y^2 * x) * (0^2 * (x * y)))) \geq \min\left\{{}^{\delta}_{\xi}((x * y) * z), {}^{\delta}_{\xi}(z)\right\} > 0.$$

Hence $x * ((y^2 * x) * (0^2 * (x * y))) \in O({}^{\delta}_{\xi})$, and therefore $O({}^{\delta}_{\xi})$ is a commutative ideal of $(X, 0)_*$. □

Theorem 3.24. *If a Łukasiewicz fuzzy set* ${}^{\delta}_{\xi}$ *in* X *satisfies* (21) *and*

$$(\forall x, y, z \in X)(\forall t_a, t_c \in (0, 1]) \left(\begin{array}{l} \langle ((x * y) * z)/t_a\rangle \in {}^{\delta}_{\xi}, \ \langle z/t_c\rangle \in {}^{\delta}_{\xi} \\ \quad \Rightarrow \ \langle (x * ((y^2 * x) * (0^2 * (x * y))))/\max\{t_a, t_c\}\rangle \, q \, {}^{\delta}_{\xi} \end{array} \right). \tag{41}$$

then the non-empty O-set of ${}^{\delta}_{\xi}$ *is a commutative ideal of* $(X, 0)_*$.

Proof. Let $O({}^{\delta}_{\xi})$ be a non-empty O-set of ${}^{\delta}_{\xi}$. Then there exists $x \in O({}^{\delta}_{\xi})$, and so $t := {}^{\delta}_{\xi}(x) > 0$, i.e., $\langle x/t\rangle \in {}^{\delta}_{\xi}$ for $t > 0$. Hence $\langle 0/t\rangle \in {}^{\delta}_{\xi}$ by (21), and thus ${}^{\delta}_{\xi}(0) \geq t > 0$. Hence $0 \in O({}^{\delta}_{\xi})$. Let $x, y, z \in X$ be such that $(x*y)*z \in O({}^{\delta}_{\xi})$ and $z \in O({}^{\delta}_{\xi})$. Then $\xi((x*y)*z) + \delta > 1$ and $\xi(z) + \delta > 1$. Since $\langle ((x*y)*z)/{}^{\delta}_{\xi}((x * y) * z)\rangle \in {}^{\delta}_{\xi}$ and $\langle z/{}^{\delta}_{\xi}(z)\rangle \in {}^{\delta}_{\xi}$, it follows from (41) that

$$\langle (x * ((y^2 * x) * (0^2 * (x * y))))/\max\{{}^{\delta}_{\xi}((x * y) * z), \, {}^{\delta}_{\xi}(z)\}\rangle \, q \, {}^{\delta}_{\xi}.$$

If $x * ((y^2 * x) * (0^2 * (x * y))) \notin O({}^{\delta}_{\xi})$, then ${}^{\delta}_{\xi}(x * ((y^2 * x) * (0^2 * (x * y)))) = 0$, and so

$$\begin{aligned} &{}^{\delta}_{\xi}(x * ((y^2 * x) * (0^2 * (x * y)))) + \max\{{}^{\delta}_{\xi}((x * y) * z), \, {}^{\delta}_{\xi}(z)\} = \max\{{}^{\delta}_{\xi}((x * y) * z), \, {}^{\delta}_{\xi}(z)\} \\ &= \max\{\max\{0, \xi((x * y) * z) + \delta - 1\}, \, \max\{0, \xi(z) + \delta - 1\}\} \\ &= \max\{\xi((x * y) * z) + \delta - 1, \, \xi(z) + \delta - 1\} \\ &= \max\{\xi((x * y) * z), \, \xi(z)\} + \delta - 1 \\ &\leq 1 + \delta - 1 \leq 1. \end{aligned}$$

Hence $\langle (x * ((y^2 * x) * (0^2 * (x * y))))/\max\{{}^{\delta}_{\xi}((x * y) * z), \, {}^{\delta}_{\xi}(z)\}\rangle \, \overline{q} \, {}^{\delta}_{\xi}$, a contradiction. Thus $x * ((y^2 * x) * (0^2 * (x * y))) \in O({}^{\delta}_{\xi})$, and therefore $O({}^{\delta}_{\xi})$ is a commutative ideal of $(X, 0)_*$. □

Theorem 3.25. *If a Łukasiewicz fuzzy set* ${}_{\xi}^{\delta}$ *in* X *satisfies* $\langle 0/\delta\rangle\, q\, \xi$ *and*

$$(\forall x, y, z \in X)\left(\begin{array}{l} \langle ((x*y)*z)/\delta\rangle\, q\, \xi,\ \langle z/\delta\rangle\, q\, \xi \\ \quad\Rightarrow\ \langle (x*((y^2*x)*(0^2*(x*y))))/\delta\rangle \in {}_{\xi}^{\delta} \end{array}\right), \tag{42}$$

then the O-set of ${}_{\xi}^{\delta}$ *is a commutative ideal of* $(X,0)_*$.

Proof. Let $O({}_{\xi}^{\delta})$ be the O-set of ${}_{\xi}^{\delta}$. If $\langle 0/\delta\rangle\, q\, \xi$, then $\xi(0)+\delta > 1$ and so

$${}_{\xi}^{\delta}(0) = \max\{0, \xi(0)+\delta-1\} = \xi(0)+\delta-1 > 0.$$

Hence $0 \in O({}_{\xi}^{\delta})$. Let $x, y, z \in X$ be such that $(x*y)*z \in O({}_{\xi}^{\delta})$ and $z \in O({}_{\xi}^{\delta})$. Then $\xi((x*y)*z)+\delta > 1$ and $\xi(z)+\delta > 1$, i.e., $\langle ((x*y)*z)/\delta\rangle\, q\, \xi$ and $\langle z/\delta\rangle\, q\, \xi$. It follows from (42) that $\langle (x*((y^2*x)*(0^2*(x*y))))/\delta\rangle \in {}_{\xi}^{\delta}$, which shows ${}_{\xi}^{\delta}(x*((y^2*x)*(0^2*(x*y)))) \geq \delta > 0$. Hence $x*((y^2*x)*(0^2*(x*y))) \in O({}_{\xi}^{\delta})$, and therefore $O({}_{\xi}^{\delta})$ is a commutative ideal of $(X,0)_*$. □

Theorem 3.26. *Let* ${}_{\xi}^{\delta}$ *be a Łukasiewicz fuzzy set in* X *that satisfies:*

$$(\forall y \in X)(\forall t \in [\delta,1])\left(\langle y/t\rangle\, q\, \xi\ \Rightarrow\ \langle 0/\delta\rangle \in {}_{\xi}^{\delta}\right), \tag{43}$$

$$(\forall x, y, z \in X)(\forall t_a, t_c \in [\delta,1])\left(\begin{array}{l} \langle ((x*y)*z)/t_a\rangle\, q\, \xi,\ \langle z/t_c\rangle\, q\, \xi \\ \quad\Rightarrow x*((y^2*x)*(0^2*(x*y))) \in ({}_{\xi}^{\delta}, \delta)_{\in} \end{array}\right). \tag{44}$$

Then the O-set of ${}_{\xi}^{\delta}$ *is a commutative ideal of* $(X,0)_*$.

Proof. Let $t \in [\delta,1]$ and $y \in O({}_{\xi}^{\delta})$. Then $\xi(y)+t \geq \xi(y)+\delta > 1$, and so $\langle y/t\rangle\, q\, \xi$, which implies that $\langle 0/\delta\rangle \in {}_{\xi}^{\delta}$ by (43). Hence ${}_{\xi}^{\delta}(0) \geq \delta > 0$, i.e., $0 \in O({}_{\xi}^{\delta})$. Let $t_a, t_c \in [\delta,1]$ and $x, y, z \in X$ be such that $\langle ((x*y)*z)/t_a\rangle\, q\, \xi$ and $\langle z/t_c\rangle\, q\, \xi$. Then $\xi((x*y)*z)+t_a \geq \xi((x*y)*z)+\delta > 1$ and $\xi(z)+t_c \geq \xi(z)+\delta > 1$. Thus $\langle ((x*y)*z)/t_a\rangle\, q\, \xi$ and $\langle z/t_c\rangle\, q\, \xi$ Using (44) leads to $x*((y^2*x)*(0^2*(x*y))) \in ({}_{\xi}^{\delta}, \delta)_{\in}$. Hence ${}_{\xi}^{\delta}(x*((y^2*x)*(0^2*(x*y)))) \geq \delta > 0$, and so $x*((y^2*x)*(0^2*(x*y))) \in O({}_{\xi}^{\delta})$. Consequently, $O({}_{\xi}^{\delta})$ is a commutative ideal of $(X,0)_*$. □

Corollary 3.27. *Let* ${}_{\xi}^{\delta}$ *be a Łukasiewicz fuzzy set in* X *that satisfies:*

$$(\forall x, y \in X)\left(\langle y/\delta\rangle\, q\, \xi\ \Rightarrow\ \langle 0/\delta\rangle \in {}_{\xi}^{\delta}\right), \tag{45}$$

$$(\forall x, y, z \in X)\left(\begin{array}{l} \langle ((x*y)*z)/\delta\rangle\, q\, \xi,\ \langle z/\delta\rangle\, q\, \xi \\ \quad\Rightarrow x*((y^2*x)*(0^2*(x*y))) \in ({}_{\xi}^{\delta}, \delta)_{\in} \end{array}\right). \tag{46}$$

Then the O-set of ${}_{\xi}^{\delta}$ *is a commutative ideal of* $(X,0)_*$.

4 Conclusion

The concept of Łukasiewicz fuzzy sets using Łukasiewicz t-norm was first introduced by Y. B. Jun, and it was applied to BCK/BCI-algebras. For the purpose of applying the Łukasiewicz fuzzy set to a commutative ideal in BCI-algebras, we introduced the concept of Łukasiewicz fuzzy commutative ideals and study its properties. We established the relationship between a Łukasiewicz fuzzy ideal and a Łukasiewicz fuzzy commutative ideal, and provided an example to show that a Łukasiewicz fuzzy ideal may not be a Łukasiewicz fuzzy commutative ideal. We explored the conditions under which a Łukasiewicz fuzzy ideal can be a Łukasiewicz fuzzy commutative ideal. We considered characterizations of Łukasiewicz fuzzy commutative ideals, and explored the conditions under which $\in$-set, q-set, and O-set can be commutative ideals.

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